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Viscosity Solutions of Hamilton–Jacobi Equations in Infinite Dimensions. V. Unbounded Linear Terms and B-Continuous Solutions*

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I. INTRODUCTION

We continue the systematic study of first-order Hamilton–Jacobi equations in infinite dimensions begun in Parts I–IV of this series [12]. Our approach relies on, as before, an appropriate interpretation of the notion of viscosity solutions as introduced in [11, 10]. In the interim, the strong development of "viscosity solution" theory in many directions has continued unabated. In addition to the many references given in earlier papers in this series (which we will not repeat here; see [12]) we would like to point out that a complete theory of second-order fully nonlinear and possibly degenerate elliptic equations in finite dimensions has been made possible by the use of viscosity solutions [8, 9, 18, 20–22, 19, 32, 33] for example) and there is now a substantial body of work concerning second order equations in infinite dimensions [25–27].

However, in this work we come back to the particular situation studied in [12, Part IV]; namely, we come back to Hamilton–Jacobi equations in

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a real separable infinite dimensional Hilbert space H involving linear terms of the form $\langle Ax, \nabla u\rangle$. Here and everywhere below, A is a linear and densely defined maximal monotone operator in H, $\langle \cdot, \cdot \rangle$ is the scalar product of H (which is identified with its dual) and ∇u corresponds to the Fréchet differential of u with respect to $x \in H$. Control problems involving the semigroup generated by $-A$ lead (see [2] and Sections IV and VII below) to the following stationary and evolutionary Hamilton–Jacobi equations:

$$u + \langle Ax, \nabla u\rangle + F(x, \nabla u) = 0 \qquad \text{in } H \tag{S}$$

and

$$u_t + \langle Ax, \nabla u\rangle + F(t, x, \nabla u) = 0 \qquad \text{in} \quad (0, T] \times H, \tag{E}$$

where the solution $u(x)$ or $u(t, x)$ is a real-valued function defined on H or on $[0, T] \times H$, $T > 0$ is given, $F(t, x, p)$ and $F(x, p)$ are real-valued and uniformly continuous on each bounded subset of $[0, T] \times H \times H$.

We will follow the approach introduced in [12, Part IV] concerning the interpretation of (S) and (E) in the viscosity sense and the structure conditions to be imposed on A and F; in particular, we interpret the term $\langle Ax, \nabla u\rangle$ via a transposition involving the (unbounded) operator A and discarding (undefined) terms of (formally) known sign due to the monotonicity of A and A^*; i.e., we use $\langle Ax, x\rangle \geqslant 0$, $\langle x, A^*x\rangle \geqslant 0$ "for all $x \in H$." The principal advance in the current work over the results of [12, Part IV] is that we no longer need to assume that solutions are weakly sequentially continuous. Indeed, we will deal with another notion of continuity called *B-continuity*. If B is a bounded linear operator in H and $u: H \to \mathbb{R}$ we say that u is B-continuous if $u(x_n) \to u(x)$ whenever $x_n \rightharpoonup x$ weakly in H and $Bx_n \to Bx$ in H. Of course, the relevant operators B will have to be chosen well with respect to the problems (S) and (E).

This seemingly small modification in fact requires drastic modifications in the proofs and significantly extends the domain of applicability of the theory. For example, the uniqueness results for (S) stated in Section III will be proved in the same section by a rather delicate use of tools from the theory of perturbed optimization (due to I. Ekeland [14], C. Stegall [31], and J. Bourgain [7]) in the completion of H in the norm $\|Bx\|$ to produce the maximum points necessary for "viscosity solution" proofs.

The existence results for (S), which are stated in Section III and proved in Section IV, will be established by a totally different argument from the one given in [12, Part IV]. Indeed, as we will see below, the existence results of [12, Part IV] correspond to the case in which B is compact; in this event B-continuity reduces to sequential weak continuity and the extra compactness properties this implies allow us to produce solutions by a kind of Faedo–Galerkin approximation. In the current generality we cannot argue in this manner since, on the one hand, we allow the case $A = 0$, $B = I$ in our analysis and on the other we have shown in [12, Part II] that Faedo–Galerkin approximations may fail to converge in this situation! For this reaon we will have to construct the solutions "by hand" via representation formulas from differential games and optimal control, thereby following the path we first took to establish existence in infinite dimensional problems in [12, Parts II and III] while relying on some ideas from G. Barles [4, 5]. This construction is performed in Section V. By the way, as in [12, Part IV], we are unable to prove existence through Perron's method as adapted to viscosity solutions by H. Ishii [16].

Basing the existence proofs on control theory and differential game considerations requires us to know that the value functions for problems in differential games and optimization are viscosity solutions of associated Hamilton–Jacobi equations (Bellman equations in the first case and Isaacs equations in the second). Difficulties arise in establishing this owing to the presence of the unbounded operator A and the rather delicate formulation of the notion of viscosity solutions. We separate this question, which is of independent interest, from the existence theory and first treat the relation between optimal control problems and viscosity solutions corresponding to (S) (i.e., infinite horizon problems) in Section IV. This is done in substantial generality as regards the behaviour of the data as x "tends to ∞" in H. In particular, we will present uniqueness results which are not contained in those presented in Section III but which in fact follow from combining the arguments in Section III and in [13]. The corresponding treatment for finite horizon problems—which correspond to (E)—will be given in Section VII.

In Section VI we formulate the main existence and uniqueness results for (E) and explain how to modify the proofs given for (S) to deal with this case. Finally, in Section VIII, we return to the question of weakly continuous solutions and consider the situation in which B is not compact but the B-continuous viscosity solutions are in fact weakly continuous. Roughly speaking, this corresponds to the situation when all the data are weakly continuous and the solution inherits this property from the data. A typical case of this sort arises in control problems corresponding to controlled parabolic or wave equations set in all of $\mathbb{R}^n$ instead of a bounded domain.

We conclude this Introduction with a few remarks. First, the notion of B-continuity (more accurately, $B^{1/2}$-continuity) is implicit in [12, Part IV] and has been used in [27] for related reasons. Next, we do not claim that this is the end of the story. Indeed, we intend to take up cases in which the nonlinearity F is itself discontinuous on H in a subsequent work. This situation arises in more realistic control problems than are covered by the current scope of the theory, including the Bellman equations which arise in control of variational inequalites, stationary *pde*'s or Navier–Stokes equations. Finally, other works treating various aspects of related problems in infinite dimensions include [1–3, 6, 29].

II. Preliminaries, Definitions, and Elementary Properties

We recall that throughout this work we assume that

$$A \text{ is a linear and densely defined maximal monotone operator in } H, \tag{A}$$

where H is a separable real Hilbert space which carries the inner-product $\langle \cdot, \cdot \rangle$ and the norm $\| \ \|$. A^* will denote the adjoint of A and e^{-tA}, e^{-tA^*} are the strongly continuous contraction semigroups generated by $-A$ and A^*.

Certain radial functions on H will play an important role here (as in [12, Part IV]) as we adopt the same notational conventions in this regard. A mapping $g: H \to \mathbb{R}$ is radial if it has the form $g(x) = h(\|x\|)$ for some $h: \mathbb{R} \to \mathbb{R}$; g is called nondecreasing if h is nondecreasing, increasing if h is increasing, ... We will hereafter abuse notation by writing $g(x) = g(\|x\|)$ so that, in particular, $g'(r) > 0$ for $r > 0$ really means $h'(r) > 0$ for $r > 0$. A radial function $g(x) = g(\|x\|)$ is differentiable at $x \in H \setminus \{0\}$ if and only if $g(r)$ is differentiable at $r = \|x\|$ and then

$$\nabla g(x) = g'(\|x\|)\hat{x}, \tag{2.1}$$

where

$$\hat{x} = \frac{x}{\|x\|} \qquad \text{for} \quad x \in H \setminus \{0\}. \tag{2.2}$$

Moreover, g is differentiable at 0 if and only if $g'(0) = 0$ and then $\nabla g(0) = 0$ and g is continuously differentiable on H exactly when $g'(r)$ is continuous and $g'(0) = 0$. We will use the notation (2.1) in this case even if $x = 0$ with the understanding that $\nabla g(0) = 0$.

We will use a variety of spaces. Let V be a locally convex vector space and Ω be a subset of V. Then

$$C(\Omega) = \{u: \Omega \to \mathbb{R};\ u \text{ is continuous}\},$$

$$C_b(\Omega) = \{u: \Omega \to \mathbb{R};\ u \text{ is continuous and bounded on bounded subsets of } \Omega\},$$

$$UC(\Omega) = \{u: \Omega \to \mathbb{R};\ u \text{ is uniformly continuous}\},$$

$$UC_s([0,T]\times\Omega)=\{u\in C(\Omega\times[0,T]); u(t,\cdot)\in UC(\Omega)$$
$$\text{uniformly in } t\in[0,T]\},$$
$$BUC(\Omega)=\{u\in UC(\Omega); u \text{ is bounded}\},$$
$$BUC_s([0,T]\times\Omega)=\{u\in UC_s([0,T]\times\Omega); u \text{ is bounded}\}.$$

Finally, if B is a positive self-adjoint, bounded operator on H and $\alpha>0$ we will denote by $H_{-\alpha}$ the Hilbert space which is the completion of H in the norm

$$\|x\|_{-\alpha}=\langle B^{\alpha}x, x\rangle^{1/2}.$$

In the natural way, $B^{\alpha/2}$ is an isometry from $H_{-\alpha}$ onto H and $D(B^{-\alpha/2})$ may be isometrically identified with the dual of $H_{-\alpha}$ using the norm

$$\|x\|_{\alpha}=\|B^{-\alpha/2}x\|^2.$$

We denote this dual by H_α; $B^{-\alpha/2}$ is an isometry from H_α onto H. In what follows, we will mainly work with H_{-1}, H_{-2}, and H_2.

We now recall from [12, Part IV] the definition of viscosity solutions of

$$\langle Ax, \nabla u\rangle+F(x,u,\nabla u)=0 \qquad \text{in } \Omega, \tag{2.3}$$

where Ω is an open set in H and $F\colon H\times\mathbb{R}\to\mathbb{R}$ is continuous. However, we shall hereafter drop the modifier "viscosity" and simply talk about "solutions."

DEFINITION 2.1. Let $u\in C(\Omega)$. Then u is a subsolution (respectively, supersolution) of (2.3) if for every $\varphi\colon\Omega\to\mathbb{R}$ with the properties

$$\left.\begin{array}{l}\varphi \text{ is weakly sequentially lower semicontinuous}\\ \text{and } \nabla\varphi \text{ and } A^*\nabla\varphi \text{ are continuous}\end{array}\right\} \tag{2.4}$$

and every $g\colon H\to\mathbb{R}$ satisfying

$$g \text{ is radial, nondecreasing, and continuously differentiable on } H \tag{2.5}$$

and local maximum (respectively, minimum) $z\in\Omega$ of $u-\varphi-g$ (respectively, $u+\varphi+g$) we have

$$\langle z, A^*\nabla\varphi(z)\rangle+F(z,u(z),\nabla\varphi(z)+\nabla g(z))\leqslant 0 \tag{2.6}$$

(respectively,

$$-\langle z, A^*\nabla\varphi(z)\rangle+F(z,u(z),-\nabla\varphi(z)-\nabla g(z))\geqslant 0). \tag{2.7}$$

Finally, u is a solution of (2.3) if it is both a subsolution and a supersolution.

The assumptions concerning $\nabla\varphi$ in (2.4) mean that the Fréchet derivative $\nabla\varphi$ of φ exists, is continuous on Ω, takes its values in $D(A^*)$ and $\nabla\varphi$, $A^*\nabla\varphi$ are continuous on Ω. Finally, it is worth recalling and emphasizing that (2.6) and (2.4) are inequalities one expects for viscosity solutions except that terms formally corresponding to the expression $\langle Ax, \nabla g(x)\rangle$ (or $\langle x, A^*\nabla g(x)\rangle$) have been dropped. The intuitive reason that this is all right is that formally

$$\langle Ax, \nabla g(x)\rangle = g'(\|x\|)\langle Ax, \hat{x}\rangle, \qquad \langle x, A^*\nabla g(x)\rangle = g'(\|x\|)\langle x, A^*\hat{x}\rangle$$

and these expressions should be nonnegative since A is maximal monotone and g is nondecreasing.

As already mentioned above, our analysis, just as in [12, Part IV] will rely on an auxiliary operator B. Concerning B we assume that, unless otherwise said,

$$\left.\begin{array}{l} B \text{ is a linear bounded positive self-adjoint operator on } H \\ \text{and } A^*B \text{ is a bounded operator on } H. \end{array}\right\} \tag{B}$$

Every closed densely defined operator A admits an operator B with the properties (B)—namely, $B=(I+AA^*)^{-1/2}$.

We will use the notations

$$x_n \to x \qquad \text{and} \qquad x_n \rightharpoonup x$$

to denote, respectively, norm and weak convergence of x_n to x and

$$B(y, r) = \{x \in H;\ |y-x| \leqslant r\}, \qquad B_R = B(0, R)$$

for all $r>0$, $R>0$.

DEFINITION 2.2. Let $\Omega \subset H$ and $u\colon \Omega \to \mathbb{R}$. Then u is B-continuous on Ω if $u(x_n) \to u(x)$ whenever x_n is a sequence in Ω, $x_n \rightharpoonup x \in \Omega$ and $Bx_n \to Bx$.

We collect some simple facts in the next lemma.

LEMMA 2.3. *Let B be an arbitrary bounded operator on H. We have:*

(i) *If u is B-continuous on H, then u is continuous on H.*

(ii) *If B is also compact, then u is B-continuous if and only if u is weakly sequentially continuous.*

(iii) *If B is positive and self-adjoint, then u is B-continuous on H if and only if u is continuous with respect to the H_{-2} norm on each ball B_R, $R>0$.*

(iv) *If B is positive and self-adjoint and $\alpha > 0$, then u is B^α-continuous on H if and only if u is B^β-continuous on H for all $\beta > 0$.*

Proof. We prove (iv), which is the only assertion which is not obvious. Since B is bounded, it is clear that B^α-continuity implies B^β-continuity for $\beta \in (0, \alpha]$. If we can also show that $B^{1/2}$-continuity implies B-continuity, the result will follow for then B^α-continuity implies $B^{2^n\alpha}$-continuity for $n = 1, 2, \ldots$ and $2^n\alpha > \beta$ if n is large.

Clearly it suffices to show that $x_n \rightharpoonup x$ and $Bx_n \to Bx$ imply $B^{1/2}x_n \to B^{1/2}x$ and for this it is enough to show that $\|B^{1/2}x_n\|^2 \to \|B^{1/2}x\|^2$. However,

$$\|B^{1/2}x_n\|^2 = \langle Bx_n, x_n\rangle \to \langle Bx, x\rangle = \|B^{1/2}x\|^2,$$

establishing the claim.

PROPOSITION 2.4. *Let u be B-continuous on H and assume that (A), (B) hold. Then the conditions* (2.5) *on g in Definition* 2.1 *may be replaced by*

$$g \text{ is radial, twice continuously differentiable on } H \text{ and } g'(r) > 0 \text{ for } r > 0 \tag{2.8}$$

without changing the notion of sub and supersolutions. Likewise, the local extremum z in (2.6) *and* (2.7) *may be required to have the property*

$$\left.\begin{array}{l}z \text{ is a local strict (in the norm topology of } H\text{) extremum in}\\ \text{the sense that there is a neighborhood } N \text{ of } z \text{ in } H \text{ such that}\\ \text{extremizing sequences chosen from } N \text{ converge strongly to } z.\end{array}\right\} \tag{2.9}$$

Remark 2.5. As is usual in the viscosity theory, many other equivalent formulations are possible. In particular, one can restrict attention to $g \in C^\infty$ and, under additional boundedness assumptions, one can assume the local extrema are global. Notice also that z is a strict local (for example) maximum point of $u - \varphi - g$ in the above sense if and only if there exists $\delta > 0$ and a positive continuous function ω on $(0, \delta]$ such that

$$u(x) - \varphi(x) - g(x) \leqslant u(z) - \varphi(z) - g(z) - \omega(\|x - z\|) \qquad \text{for all} \quad x \in B(z, \delta). \tag{2.10}$$

Remark 2.6. In view of Proposition 2.4, the observations made in [12, Part IV, Appendix] concerning the possibility of "replacing 0 by $x_0 \in D(A^*)$" in Definition 2.1 remain valid for B-continuous solutions in place of weakly continuous solutions.

Proof of Proposition 2.4. Since B-continuous functions are continuous, the fact that we may restrict attention to strict local extrema satisfying (2.10) (and hence (2.9)) has been established in [12, Part IV].

The proof of the claim concerning (2.8) provides a typical example of the perturbation arguments needed to work with B-continuous solutions.

Let, for example, u be B-continuous, g and φ have the properties of Definition 2.1 and z be a maximum of $u-\varphi-g$ for which (2.10) holds with $\omega>0$ on $(0, \delta]$ (which we may assume, without loss of generality, by the above). Assuming that then (2.6) would hold if g satisfied (2.8), we choose a sequence g_n of radial functions satisfying (2.8) and

$$g_n \to g, \quad g_n' \to g' \qquad \text{uniformly on} \quad [0, \|z\|+\delta].$$

Now we would like to maximize

$$\Phi_n(x)=u(x)-\varphi(x)-g_n(x)$$

on $B(z, \delta)$. This is where weak continuity was employed in [12, Part IV] and where here we invoke the B-continuity. $B(z, \delta)$ is a closed convex set in H_{-2} and Φ_n is weakly sequentially upper-semicontinuous on $B(z, \delta)$ in the H_{-2} topology because u is B-continuous while φ and g are weakly lower-semicontinuous on H and bounded on $B(z, \delta)$. Therefore, by the perturbed optimization results due to I. Ekeland [14], C. Stegall [31], and J. Bourgain [7], for each $\varepsilon>0$ there exists $p_\varepsilon \in H_2$ such that

$$\|p_\varepsilon\|_2 \leqslant \varepsilon \text{ and } \Phi_n(x)+\langle p_\varepsilon, x\rangle \text{ admits a maximum point } z_n^\varepsilon \text{ on } B(z, \delta).$$

Since B defines an isometry of H onto $H_2=D(B^{-1})$, we may write $p_\varepsilon=Bq_\varepsilon$, where $q_\varepsilon \in H$ and $\|q_\varepsilon\| \leqslant \varepsilon$. Next we observe that putting

$$\delta_n=\sup_{[0, \|z\|+\delta]} |g_n-g|, \qquad C_0=(\|z\|+\delta)\,\|B\|$$

we have $\delta_n \to 0$ and

$$\begin{aligned}
\Phi(z_n^\varepsilon) &\geqslant \Phi_n(z_n^\varepsilon)-\delta_n\\
&\geqslant \Phi_n(z_n^\varepsilon)+(Bq_\varepsilon, z_n^\varepsilon)-\delta_n-C_0\varepsilon\\
&\geqslant \Phi_n(z)+\langle Bq_\varepsilon, z\rangle-\delta_n-C_0\varepsilon\\
&\geqslant \Phi(z)-2\delta_n-2C_0\varepsilon.
\end{aligned}$$

Therefore, in view of (2.10), $z_n^\varepsilon \to z$ in H as $\varepsilon \downarrow 0$, $n \to \infty$. In particular, for ε small enough and n large enough, z_n^ε is a local maximum point of $u(x)-(\varphi(x)-\langle q_\varepsilon, Bx\rangle)-g_n(x)$ and then, by assumption, we have

$$\langle z_n^\varepsilon, A^*(\nabla\varphi(z_n^\varepsilon)-Bq_\varepsilon)\rangle+F(z_n^\varepsilon, u(z_n^\varepsilon), \nabla\varphi(z_n^\varepsilon)-Bq_\varepsilon+\nabla g_n(z_n^\varepsilon)) \leqslant 0.$$

Letting $\varepsilon \downarrow 0$, and $n \to \infty$ we conclude that (2.6) indeed holds. The case of supersolutions is treated analogously.

Using similar arguments, we may also prove a "stability-consistency" result on sequences of B-continuous solutions.

PROPOSITION 2.7. *Let (A), (B) hold and let u_n be a sequence of B-continuous subsolutions (respectively, supersolutions) of*

$$\langle Ax, \nabla u_n \rangle + F_n(x, u_n, \nabla u_n) = 0 \qquad \text{on } H, \tag{2.11}$$

where $F_n \in C(H \times \mathbb{R} \times H)$. Let $F \in C(H \times \mathbb{R} \times H)$, u be B-continuous and assume

$$\text{for all } x \in H \text{ there is a } \delta > 0 \text{ such that } u_n \to u \text{ uniformly on } B(x, \delta) \tag{2.12}$$

and

$$F_n(x_n, r_n, p_n) \to F(x, r, p) \qquad \text{if} \quad x_n \to x,\ r_n \to r \text{ and } p_n \to p. \tag{2.13}$$

Then u is a subsolution (respectively, supersolution) of (2.3) on H.

Proof. We discuss the case of subsolutions. Let z be a strict maximum point of $u - \varphi - g$ with φ, g as above. Slight modifications of the arguments which established Proposition 2.4 show that for sufficiently small $\varepsilon > 0$ and sufficiently large n and a suitable $q_n^\varepsilon \in H$ with $\|q_n^\varepsilon\| \leqslant \varepsilon$ there is a local maximum point z_n^ε of $u_n - \varphi - g + \langle Bq_n^\varepsilon, x \rangle$ satisfying $z_n^\varepsilon \to z$ as $\varepsilon \downarrow 0$ and $n \to \infty$. By assumption

$$\langle z_n^\varepsilon, A^*(\nabla\varphi(z_n^\varepsilon) - Bq_n^\varepsilon)\rangle + F(z_n^\varepsilon, u_n(z_n^\varepsilon), \nabla\varphi(z_n^\varepsilon) - Bq_n^\varepsilon + \nabla g_n(z_n^\varepsilon)) \leqslant 0$$

and passing to the limit, we conclude.

We end this section with some remarks about (E). First, while (S) is obviously included in (2.3), this is not so for (E). For (E) we replace H by $\hat{H} = H \times \mathbb{R}$, A by the operator $\hat{A}(x, t) = (Ax, 0)$ on $D(\hat{A}) = D(A) \times \mathbb{R}$ in $\hat{H}$ and B by $\hat{B}(x, t) = (Bx, t)$ on $\hat{H}$. Clearly $\hat{B}$ is self-adjoint, bounded, positive, and $\hat{A}^*\hat{B}$ is bounded on $\hat{H}$. If $u\colon H \times [0, T] \to \mathbb{R}$, we will say that u is B-continuous if u is $\hat{B}$-continuous on $H \times [0, T]$, i.e., if

$$u(x_n, t_n) \to u(x, t) \qquad \text{whenever} \quad x_n \rightharpoonup x \text{ in } H$$

and

$$(Bx_n, t_n) \to (Bx, t) \qquad \text{in} \quad H \times [0, T].$$

Using Remark 2.6 and the arguments given in [12, Part IV, Appendix] we immediately obtain

PROPOSITION 2.8. *Let* (A) *and* (B) *hold,* $F \in C([0,T] \times H \times H)$, u *be* B*-continuous on* $H \times [0,T]$ *and a subsolution* (*respectively, supersolution*) *of* E *on* $H \times (0,T)$. *Then for every* $\varphi\colon [0,T] \times H \to \mathbb{R}$ *satisfying*

$$\left.\begin{array}{l}\varphi \text{ is weakly lower-semicontinuous on } (0,T] \times H \\ \nabla\varphi \text{ and } A^*\nabla\varphi \text{ are continuous on } [0,T] \times H\end{array}\right\} \tag{2.14}$$

and every g *satisfying* (2.8) *and every local maximum* (*respectively, minimum*) $(t,z) \in (0,T] \times H$ *of* $u - \varphi - g$ (*respectively,* $u + \varphi + g$) *on* $H \times [0,T]$ *we have*

$$\frac{\partial\varphi}{\partial t}(t,z) + \langle z, A^*\nabla\varphi(z,t)\rangle + F(t,z,\nabla\varphi(t,z) + \nabla g(z)) \leqslant 0 \tag{2.15}$$

(*respectively,*

$$-\frac{\partial\varphi}{\partial t}(t,z) - \langle z, A^*\nabla\varphi(z,t)\rangle + F(t,z,-\nabla\varphi(t,z) - \nabla g(z)) \geqslant 0). \tag{2.16}$$

Remark 2.9. Just as in Proposition 2.4 and Remark 2.5, we may restrict our attention to smooth choices of g and strict extrema, etc., without changing the set of B-continuous functions u for which (2.15) and (2.16) hold under the above assumptions.

We also point out that the weak lower-semicontinuity of φ in Definition 2.1 could be replaced by the assumption that φ is B-lower-semicontinuous throughout this work.

We conclude by mentioning that in all that follows we will assume that

$$F \text{ is uniformly continuous on bounded sets} \tag{2.17}$$

(in $H \times H$ for (S) and in $[0,T] \times H \times H$ for (E)). In addition, it will always be assumed that (A) and (B) hold.

III. EXISTENCE AND UNIQUENESS RESULTS FOR (S); UNIQUENESS PROOFS

We begin with the problem (S). We will always assume (A), (B), and (2.17). In addition, we will employ the following structure assumption on F

$$\begin{aligned} &F(t,y,\lambda B(x-y)) - F(t,x,\lambda B(x-y)) \\ &\qquad \leqslant \omega(\|x-y\|\,(1 + \lambda\,\|x-y\|_{-1})) \end{aligned} \tag{3.1}$$

for all $x, y \in H$, $t \in [0,T]$, $\lambda \geqslant 0$, where $\omega(r) \to 0$ as $r \downarrow 0$ (and we may assume ω to be continuous, increasing, and subadditive on $[0,\infty)$.

The other assumptions involve functions $\mu, \nu\colon H \to [0, \infty)$ such that

$$\left.\begin{array}{l}\mu \text{ and } \nu \text{ are Lipschitz continuous, radial } C^1\text{, nondecreasing, nonnegative,}\\ \text{and } \lim_{\|x\| \to \infty} \mu(x) = \infty \text{ and } \liminf_{\|x\| \to \infty} \nu(x)/\|x\| \geqslant 1;\end{array}\right\} \tag{3.2}$$

we will assume that

$$\begin{aligned}\max(F(t, x, p) - F(t, x, p + \lambda\nabla\mu(x)), F(t, x, p - \lambda\nabla\mu(x))\\ - F(t, x, p)) \leqslant \sigma(\lambda, \|p\|)\end{aligned} \tag{3.3}$$

for $\lambda \geqslant 0$ and $x, p \in H$, where $\sigma(r, R)$ is nondecreasing in R and satisfies $\sigma(0+, R) = 0$ for $R \geqslant 0$. Sometimes we will also require

$$\begin{aligned}\min(F(t, x, p + \lambda\nabla\nu(x)) - F(t, x, p), F(t, x, p)\\ - F(t, x, p - \lambda\nabla\nu(x))) \geqslant -C_R\end{aligned} \tag{3.4}$$

for $t \in [0, T]$ $x, p \in H$, $\|p\| \leqslant R$, $0 \leqslant \lambda \leqslant R$, where C_R is a constant depending only on $R \geqslant 0$.

Exactly as in [12, Part IV], we will also need some structure conditions on A; namely, either

$$\exists C_0 \in \mathbb{R} \text{ such that for } x \in H \langle (A^*B + C_0B)x, x\rangle \geqslant \|x\|^2 \tag{3.5}$$

or

$$\exists C_0 \in \mathbb{R} \text{ such that for } x \in H \langle (A^*B + C_0B)x, x\rangle \geqslant 0 \tag*{$(3.5)_w$}$$

for all $x \in H$ and we will refer to these conditions as, respectively, the strong B condition and the weak B condition. The reader can refer to [12, Part IV] to see how these conditions may be checked and how, roughly speaking, the strong B condition corresponds to second order uniformly elliptic operators while the weak B condition correponds to wave operators.

We are now able to present our main existence and uniqueness results for (S); the first result below corresponds to the strong B case and the second to the weak B case.

THEOREM 3.1. *Assume* (3.1), (3.3), *and* (3.5).

Comparison. Let $u, v \in UC(H)$ *be* B*-continuous and, respectively, a subsolution and a supersolution of* (S) *in* H. *If either* u *and* $-v$ *are bounded from above or* (3.4) *holds, then* $u \leqslant v$.

Existence. (i) *If* $F(\cdot, 0)$ *is bounded on* H, *then* (S) *has a unique* B*-continuous solution* $u \in BUC(H)$. *Furthermore,* u *extends by continuity to* H_{-1} *and* (*so extended*) $u \in BUC(H_{-1})$.

(ii) *If* (3.4) *holds and there is a nondecreasing function* $G\colon \mathbb{R} \to \mathbb{R}$ *such that* $w(x, y) = G(\|x-y\|)$ *is an everywhere differentiable Lipschitz continuous solution of*

$$w(x, y) + F(x, \nabla_x w) - F(y, -\nabla_y w) \geqslant 0 \qquad on \quad H \times H, \tag{3.6}$$

then there is a unique B-continuous solution $u \in UC(H)$ *of* (*S*). *Furthermore, u extends by continuity to* H_{-1} *and* (*so extended*) $u \in UC(H_{-1})$.

Remark 3.2. Hereafter, when we have a function $u\colon H \to \mathbb{R}$ which extends uniquely by continuity to a continuous function on all of H_{-1}, we will simply identify u and its extension and write $u \in UC(H_{-1})$ or $u \in BUC(H_{-1})$, etc., as appropriate. We recall that, by the above, such a function u is automatically B-continuous.

Remark 3.3. The comparison assertion holds for a subsolution u and a supersolution v in an open bounded subset Ω of H provided u, v are uniformly continuous on $\bar{\Omega}$ in the H_{-1} topology and $u \leqslant v$ on $\partial\Omega$ (and then we do not need (3.3), (3.4)).

Remark 3.4. If $\langle Ax, x\rangle \geqslant \alpha \|x\|^2$ for some $\alpha > 0$ and $x \in D(A)$, then we may replace (3.6) by

$$\begin{aligned} w(x, y) + \alpha\langle x, \nabla_x w\rangle + \alpha\langle y, \nabla_y w\rangle + F(x, \nabla_x w)& \\ - F(y, -\nabla_y w) \geqslant 0 \qquad \text{on} \quad H \times H.& \end{aligned} \tag{3.7}$$

Similarly, we may replace (3.3) and (3.4) by

$$\begin{cases} F(t, x, p) - F(t, x, p + \lambda\nabla\mu(x)) - \alpha\lambda\langle x, \nabla\mu(x)\rangle \leqslant \sigma(\lambda, \|p\|), \\ F(t, x, p - \lambda\nabla\mu(x)) - F(t, x, p) - \alpha\lambda\langle x, \nabla\mu(x)\rangle) \leqslant \sigma(\lambda, \|p\|), \end{cases}$$

and

$$\begin{cases} F(t, x, p + \lambda\nabla v(x)) - F(t, x, p) + \alpha\lambda\langle x, \nabla v(x)\rangle \geqslant -C_R, \\ F(t, x, p) - F(t, x, p - \lambda\nabla v(x)) + \alpha\lambda\langle x, \nabla v(x)\rangle \geqslant -C_R \end{cases}$$

Remark 3.5. Most of the results proven in [13] for finite dimensional problems extend to (S) by combining the arguments of [13] and those presented here.

Remark 3.6. It can be shown, in the course of establishing the existence results, that the solution u we construct in fact satisfies (S) in a slightly stronger sense than the one given in Definition 1. Indeed, if $x_0 \in D(A)$ is arbitrary, φ, g satisfy the conditions of Definition 2.1 and z is a local maximum point of $u(x) - \varphi(x) - g(\|x - x_0\|)$, then

$$\langle z, A^*\nabla\varphi(z)\rangle + \langle Ax_0, \nabla g(z - x_0)\rangle + F(z, \nabla\varphi(z) + \nabla g(z - x_0)) \leqslant 0; \tag{3.8}$$

a similar strengthening of the supersolution property also holds. In [12, Part IV] it was shown that the additional inequalities (3.8) held for weakly continuous subsolutions provided that A is self-adjoint. This remains true for B-continuous solutions. It is not yet known if this is the case for general maximal monotone A.

We now turn to the weak B case, where we will replace (3.1) by

$$\begin{aligned} &F(t, y, \lambda B(x-y)) - F(t, x, \lambda B(x-y)) \\ &\quad \leqslant \omega(\|x-y\|_{-1} + \lambda \|x-y\|_{-1}^2) \end{aligned} \tag{3.9}$$

for all $x, y \in H$, $t \in [0, T]$, $\lambda \geqslant 0$, where $\omega\colon \mathbb{R} \to [0, \infty)$ is continuous, subadditive and $\omega(0) = 0$.

THEOREM 3.7. *Let* $(3.5)_w$, (3.9), *and* (3.3) *hold.*

Comparison for (S). *Let* $u, v \in UC(H_{-1})$ *be, respectively, a subsolution and a supersolution of* (S) *in* H. *If either* u *and* $-v$ *are bounded above or* (3.4) *holds, then* $u \leqslant v$ *in* H.

Existence for (S). (i) *If* $F(\cdot, 0)$ *is bounded on* H, *then there is a unique solution* $u \in BUC(H_{-1})$ of (S).

(ii) *If* (3.4) *holds and there is a nondecreasing function* $G\colon \mathbb{R} \mapsto \mathbb{R}$ *such that* $w(x, y) = G(\|x-y\|_{-1})$ *is an everywhere differentiable Lipschitz continuous solution of*

$$w(x, y) - C_0(\langle x, \nabla_x w\rangle + \langle y, \nabla_y w\rangle) + F(x, \nabla_x w) - F(y, -\nabla_y w) \geqslant 0 \tag{3.10}$$

on $H \times H$, *then there is a unique solution* $u \in UC(H_{-1})$ *of* (S).

Remark 3.8. The analogues of Remarks 3.3 and 3.4 also hold in this case.

Remark 3.9. If we compare Theorem 3.1 with [12, Part IV, Theorem 1.2] we see that we have "merely" replaced weak continuity with "B-continuity" and that in the existence assertions we have suppressed the compactness condition on B. Similarly, comparing Theorem 3.7 and [12, Part IV, Theorem 1.4] we see that we have "merely" supressed the weak continuity of solutions and the compactness of B previously required for existence (note that the B-continuity is implied by continuity in H_{-1}).

We turn now to the proofs of the comparison assertions of Theorems 3.1 and 3.7. In fact, these proofs follow the arguments in [12, Part IV] closely, so we will just explain the modifications of the proof of Theorem 1.2 of [12, Part IV] required to establish Theorem 3.1 in the case in which u and $-v$ are bounded from above. Thus we choose ε, $\lambda > 0$ and consider the function

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{2\varepsilon} \langle B(x-y), x-y \rangle - \lambda(\mu(x) + \mu(y)) \tag{3.11}$$

for $x, y \in H$, where μ is from (3.2). Clearly, $\Phi \to -\infty$ if $\|(x, y)\| \to \infty$ since u and $-v$ are bounded from above. Therefore, there is an $R > 0$ depending on λ such that

$$\Phi(x, y) \leqslant -1 \qquad \text{if} \quad \|(x, y)\| \geqslant R - 1 \tag{3.12}$$

while we may always assume that

$$\Phi(x_0, y_0) \geqslant 0 \qquad \text{for some} \quad (x_0, y_0) \in H \times H \tag{3.13}$$

since otherwise there is nothing to prove.

Arguing as in the preceding section we conclude that for all $\alpha > 0$ there are $p, q \in H$ such that $\|(p, q)\| = (\|p\|^2 + \|q\|^2)^{1/2} \leqslant \alpha$ and

$$\begin{aligned} &\Phi(x, y) + \langle Bp, x \rangle + \langle Bq, y \rangle\rangle \text{ attains a maximum over} \\ &\{(x, y) : \|(x, y)\| \leqslant R\} \text{ at } (\hat{x}, \hat{y}); \end{aligned} \tag{3.14}$$

this is the point at which we require the B-continuity of u and v. We then observe that we have

$$\begin{aligned} \Phi(\hat{x}, \hat{y}) &\geqslant \Phi(x_0, y_0) + \langle Bp, x_0 - \hat{x} \rangle + \langle Bq, y_0 - \hat{y} \rangle \\ &\geqslant -2 \|B\| \alpha R \\ &\geqslant \sup_{R-1 \leqslant \|(x, y)\| \leqslant R} (\Phi(x, y) + \langle Bp, x \rangle + \langle Bq, y \rangle) + 1 - 3 \|B\| \alpha R. \end{aligned}$$

Therefore, if $\alpha \in (0, 1/3 \|B\| R)$ we conclude that $\|(\hat{x}, \hat{y})\| < R - 1$ and thus $(\hat{x}, \hat{y})$ is a local maximum of $\hat{\Phi} = \Phi + \langle Bp, x \rangle + \langle Bq, y \rangle$.

At this stage we may copy the proof of Theorem 1.2 in [12, Part IV] provided only that we track the contributions of the terms Bp and Bq in the process. Indeed, if m is a modulus (that is, an increasing, continuous, and subadditive function on $[0, \infty)$ vanishing at 0) for which

$$|u(x) - u(y)|, \qquad |v(x) - v(y)| \leqslant m(\|x - y\|) \tag{3.15}$$

the relation $\hat{\Phi}(\hat{x}, \hat{y}) \geqslant \hat{\Phi}(\hat{x}\, \hat{x})$, $\hat{\Phi}(\hat{y}, \hat{y})$ yields

$$\frac{1}{2\varepsilon} \langle B(\hat{x} - \hat{y}), \hat{x} - \hat{y} \rangle \leqslant 2 \|B\| \alpha R + m(\|\hat{x} - \hat{y}\|). \tag{3.16}$$

Then, using the definition of viscosity solutions and (3.3), we find

$$\begin{aligned}
u(\hat{x}) &+ \left\langle \hat{x}, \frac{1}{\varepsilon} A^*B(\hat{x}-\hat{y}) - A^*Bp \right\rangle \\
&\leqslant -F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon}\right) + F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon}\right) \\
&\quad - F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon} + \lambda\nabla\mu(\hat{x})\right) + F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon} + \lambda\nabla\mu(\hat{x})\right) \\
&\quad - F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon} + \lambda\nabla\mu(\hat{x}) - Bp\right)
\end{aligned}$$

which implies

$$\begin{aligned}
u(\hat{x}) + \left\langle \hat{x}, \frac{1}{\varepsilon} A^*B(\hat{x}-\hat{y}) \right\rangle &\leqslant -F\left(\hat{x}, \frac{B(\hat{x}-\hat{y})}{\varepsilon}\right) \\
&\quad + \sigma\left(\lambda, \frac{\|B(\hat{x}-\hat{y})\|}{\varepsilon}\right) + \delta_{\varepsilon,\lambda}(\alpha),
\end{aligned}$$

where $\delta_{\varepsilon,\lambda}(\alpha)$ stands for a quantity satisfying $\delta_{\varepsilon,\lambda}(\alpha) \to 0$ as $\alpha \to 0$ for fixed $\varepsilon, \lambda > 0$. This conclusion uses the bound $\|(\hat{x}, \hat{y})\| \leqslant R-1$, and the uniform continuity of F on bounded sets. Similarly

$$\begin{aligned}
-v(y) + \left\langle \hat{y}, \frac{1}{\varepsilon} A^*B(\hat{x}-\hat{y}) \right\rangle &\leqslant F\left(\hat{y}, \frac{B(\hat{x}-\hat{y})}{\varepsilon}\right) \\
&\quad + \sigma\left(\lambda, \frac{\|B(\hat{x}-\hat{y}\|}{\varepsilon}\right) + \delta_{\varepsilon,\lambda}(\alpha).
\end{aligned}$$

Adding these two inequalities and using (3.1) and (3.5) yields

$$\begin{aligned}
u(\hat{x}) - v(\hat{y}) &\leqslant 2\delta_{\varepsilon,\lambda}(\alpha) + 2\sigma\left(\lambda, \frac{1}{\varepsilon}\|B(\hat{x}-\hat{y})\|\right) \\
&\quad + \omega\left(\|\hat{x}-\hat{y}\|\left(1 + \frac{1}{\varepsilon}\langle B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle^{1/2}\right)\right) \\
&\quad - \frac{1}{\varepsilon}\left[\langle A^*B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle + C_0\langle B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle\right] \\
&\quad + \frac{C_0}{\varepsilon}\langle B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle
\end{aligned}$$

$$\leqslant 2\delta_{\varepsilon,\lambda}(\alpha)+2\sigma\left(\lambda, \frac{1}{\varepsilon}\|B(\hat{x}-\hat{y})\|\right)$$

$$+\omega\left(\|\hat{x}-\hat{y}\|\left(1+\frac{1}{\varepsilon}\langle B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle^{1/2}\right)\right)$$

$$-\frac{1}{\varepsilon}\|\hat{x}-\hat{y}\|^2+\frac{C_0}{\varepsilon}\langle B(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle.$$

We now recall (3.16) and use the subadditivity of ω to obtain

$$u(\hat{x})-v(\hat{y})\leqslant 2\hat{\delta}_{\varepsilon,\lambda}(\alpha)+2\sigma\left(\lambda, \frac{1}{\varepsilon}\|B(\hat{x}-\hat{y})\|\right)$$

$$+\omega\left(r\left(1+\left(\frac{4\|B\|\,\alpha R+2m(r)}{\varepsilon}\right)^{1/2}\right)\right)-\frac{1}{\varepsilon}r^2+2C_0 m(r),$$

where $r=\|\hat{x}-\hat{y}\|$ and $\hat{\delta}_{\varepsilon,\lambda}(\alpha)$ has the same character as $\delta_{\varepsilon,\lambda}(\alpha)$.

As in [12, Part IV] the sum of the last three terms on the right-hand side can be bounded above by some function $\kappa(\varepsilon)$, where $\kappa(0^+)=0$. Therefore, if we can obtain a bound of the form

$$\frac{1}{\varepsilon}\|B(\hat{x}-\hat{y})\|\leqslant C_\varepsilon, \tag{3.17}$$

where C_ε is independent of α, $\lambda\in(0,1]$, we can conclude (as in [12, Part IV]) by taking the iterated limit $\alpha\to 0^+$, then $\lambda\to 0^+$, and then $\varepsilon\to 0^+$. The bound (3.17) follows from the fact since $\|(\hat{x}, \hat{y})\|\leqslant R-1$, for unit vectors w we have

$$\hat{\Phi}(\hat{x}+w, \hat{y})\leqslant\hat{\Phi}(\hat{x}, \hat{y})$$

which amounts to

$$\frac{1}{\varepsilon}\langle w, B(\hat{x}-\hat{y})\rangle\leqslant\frac{1}{\varepsilon}\langle Bw, w\rangle+u(\hat{x})-u(\hat{x}+w)$$

$$-\langle Bp, w\rangle+\lambda(\mu(\hat{x})-\mu(\hat{x}+w))$$

$$\leqslant\frac{\|B\|}{\varepsilon}+m(1)+\lambda L_\mu+\|B\|\,\alpha,$$

where L_μ is a Lipschitz constant for μ. Therefore

$$\frac{1}{\varepsilon}\|B(\hat{x}-\hat{y})\|\leqslant\frac{1}{\varepsilon}\|B\|+m(1)+\lambda L_\mu+\|B\|\,\alpha.$$

This concludes the proof of Theorem 3.1.

IV. Optimal Control Problems

In this section we consider infinite horizon optimal control problems corresponding to evolution equations involving the operator A and show that the value functions are the unique viscosity solutions of the corresponding Hamilton–Jacobi equations. Recall that the conditions (A) and (B) are always in force.

We start by defining the class of optimal control problems of interest. More precisely, let $\mathscr{A}$ be an arbitrary metric space; a control will be a measurable function from $[0, \infty)$ to $\mathscr{A}$ and a typical control will be denoted by α_t. The state X_t of the system will be given for $t \geq 0$ by the solution of

$$\frac{d}{dt} X_t + AX_t = b(X_t, \alpha_t), \qquad X_0 = x, \tag{4.1}$$

where b is continuous on $H \times \mathscr{A}$ and where the initial state is an arbitrary point $x \in H$. It is assumed that b satisfies

$$|b(x, \alpha) - b(y, \alpha)| \leq C_0 \, |x-y| \qquad \text{for} \quad x, y \in H \quad \text{and} \quad \alpha \in \mathscr{A} \tag{4.2}$$

and

$$|b(0, \alpha)| \leq C_1 \qquad \text{for} \quad \alpha \in \mathscr{A}. \tag{4.3}$$

In view of these assumptions, (4.1) has a unique (mild) solution $X_t \in C([0, \infty); H)$ which satisfies

$$\|X_t\| \leq e^{C_0 t} \, \|x\| + C_1 C_0^{-1}(e^{C_0 t} - 1) \qquad \text{for} \quad t \geq 0. \tag{4.4}$$

Moreover, if

$$|b(x, \alpha)| \leq C_2 \qquad \text{for} \quad \alpha \in \mathscr{A} \quad \text{and} \quad x \in H, \tag{4.5}$$

then

$$\|X_t\| \leq \|x\| + C_2 t \qquad \text{for} \quad t \geq 0. \tag{4.6}$$

Given a continuous function f on $H \times \mathscr{A}$ and $\lambda > 0$, one then defines an associated cost functional J depending on x and α_t by

$$J(x, \alpha_t) = \int_0^\infty f(X_t, \alpha_t) e^{-\lambda t} \, dt. \tag{4.7}$$

In view of (4.4), this integral will be convergent if

$$\exists C > 0, \quad m \in \left[0, \frac{\lambda}{C_0}\right) \qquad \text{such that} \quad |f(x, \alpha)| \leq C(1 + \|x\|)^m; \tag{4.8}$$

alternatively, if we assume (4.5), it suffices to have

$$\exists C>0, \quad m\in\left[0, \frac{\lambda}{C_2}\right) \qquad \text{such that} \quad |f(x,\alpha)|\leqslant Ce^{m\|x\|}. \tag{4.9}$$

In either case, the value function u is then defined by

$$u(x)=\inf_{\alpha_t} J(x,\alpha_t), \tag{4.10}$$

where the infimum is taken over all possible controls. Other restrictions on f which we will sometimes invoke are

$$|f(x,\alpha)|\leqslant C \qquad \text{for} \quad x\in H, \quad \alpha\in\mathscr{A}, \tag{4.11}$$

$$|f(x,\alpha)-f(y,\alpha)|\leqslant\omega(\|x-y\|) \qquad \text{for} \quad x, y\neq H, \quad \alpha\in\mathscr{A}, \tag{4.12}$$

for some modulus ω, and

$$|f(x,\alpha)-f(y,\alpha)|\leqslant\omega(\|x-y\|, R) \qquad \text{for} \quad \|x\|, \|y\|\leqslant R, \quad R>0, \quad \alpha\in\mathscr{A} \tag{4.13}$$

for some continuous function $\omega(r,s)$ which is nondecreasing in both variables, subadditive in r and satisfies $\omega(0,R)=0$ for $R\geqslant 0$ (observe that such a function exists if $f(\cdot,\alpha)$ is uniformly continuous on bounded sets uniformly in α).

The Hamilton–Jacobi equation associated with (4.10) is

$$\lambda u(x)+\langle Ax, \nabla u\rangle+\sup_{\alpha\in\mathscr{A}}[-\langle b(x,\alpha), \nabla u\rangle-f(x,\alpha)]=0 \tag{4.14}$$

and we begin by establishing that (4.10) provides viscosity solutions of (4.14).

THEOREM 4.1. *Let* (4.2) *hold and let u be given by* (4.10).

(i) (*The case of bounded evolution*). *Assume, in addition, that* (4.5), (4.11), *and* (4.12) (*respectively,* (4.5), (4.9), *and* (4.13)) *hold. Then* $u\in BUC(H)$ (*respectively, u is uniformly continuous on bounded sets and satisfies*

$$|u(x)|\leqslant Ke^{\kappa\|x\|} \qquad \textit{for} \quad x\in H, \tag{4.15}$$

where $K=C(\lambda-mC_2)^{-1}$, $\kappa=m$). *Moreover, u is a viscosity solution of* (4.14).

(ii) (*Unbounded evolution case*). *Assume, in addition to* (4.2), *that* (4.3), (4.8), *and* (4.13) *hold. Then u is uniformly continuous on bounded sets and there is a* $K\geqslant 0$ *such that*

$$|u(x)|\leqslant K(1+\|x\|)^{\kappa} \tag{4.16}$$

for $\kappa=m$. *Moreover, u is a viscosity solution of* (4.14).

Remark 4.2. The proofs will show that the fact u is a viscosity solution of (4.14) is extremely general. Roughly speaking, any assumptions which imply that u is well defined and continuous on H should imply that u is a viscosity solution. Moreover, the proofs also show that u satisfies Eq. (4.14) in the stronger sense of Remark 3.6.

Proof of Theorem 4.1. We adapt arguments from [23]. The bounds (4.15), (4.16) on u follow from (4.4), (4.6) and the corresponding bounds on f. Next we observe that (4.2) implies that if X_t^1, X_t^2 are solutions of (4.1) corresponding to initial states x^1, $x^2 \in H$ with the same control α_t, then we have

$$\|X_t^1 - X_t^2\| \leqslant e^{C_0 t} \|x^1 - x^2\| \qquad \text{for} \quad t \geqslant 0. \tag{4.17}$$

Hence, in all cases, for $T > 0$ we find

$$\begin{aligned} |u(x^1) - u(x^2)| \leqslant & \sup_{\alpha_t} \int_0^T |f(X_t^1, \alpha_t) - f(X_t^2, \alpha_t)| \, e^{-\lambda t} \, dt \\ & + \sup_{\alpha_t} \int_T^\infty (|f(X_t^1, \alpha_t)| + |f(X_t^2, \alpha_t)|) e^{-\lambda t} \, dt. \end{aligned} \tag{4.18}$$

Therefore, if (4.11), (4.12) hold, and m is a modulus for $f(\cdot, \alpha)$ uniform in $\alpha \in \mathscr{A}$, (4.17) implies

$$|u(x^1) - u(x^2)| \leqslant \int_0^T m(e^{C_0 t} \|x^1 - x^2\|) e^{-\lambda t} \, dt + 2Ce^{-\lambda T}.$$

Since $T > 0$ is arbitrary, it follows that u is uniformly continuous. On the other hand, if (4.5), (4.9), and (4.13) hold, we first observe that

$$|X_t| \leqslant R_1 = R + C_2 T \qquad \text{for} \quad t \in [0, T] \quad \text{and} \quad x \in B_R$$

and therefore for $x^1, x^2 \in B_R$, (4.18) yields

$$\begin{aligned} |u(x^1) - u(x^2)| & \leqslant \int_0^T \omega(e^{C_0 t} \|x^1 - x^2\|, R_1) e^{-\lambda t} \, dt \\ & \quad + \int_T^\infty 2Ce^{-\lambda t} e^{mC_2(t + R)} \, dt \\ & = \int_0^T \omega(e^{C_0 t} \|x^1 - x^2\|, R_1) e^{-\lambda t} \, dt + C' e^{-(\lambda - mC_2) T} \end{aligned}$$

for some positive constant C' independent of $T \in (0, \infty)$. This inequality shows the uniform continuity of u on bounded sets. Finally, if we assume (4.3), (4.8), and (4.13), we first observe that

$$\|X_t\| \leqslant R_1 = e^{C_0 T} R + C_1 C_0^{-1}(e^{C_0 T} - 1) \qquad \text{for} \quad t \in [0, T], \quad x \in B_R,$$

and if $x^1, x^2 \in B_R$, (4.18) yields

$$|u(x^1) - u(x^2)| \leqslant \int_0^T \omega(e^{C_0 t} \|x^1 - x^2\|, R_1) e^{-\lambda t}\, dt + C' e^{-(\lambda - m C_0) T}$$

exactly as above. Again, this establishes the uniform continuity of u on bounded sets.

We next have to show, in all cases, that the value function is a viscosity solution of (4.14). To this end, we consider a global minimum $\bar{x}$ of $u + \varphi + g$, where φ, g satisfy (2.4), (2.5) and show that (2.7) holds. The proof of (2.6) is similar but simpler and is omitted. The optimality principle of the dynamic programming argument states

$$u(x) = \inf_{\alpha_t} \left\{ \int_0^h f(X_t, \alpha_t) e^{-\lambda t}\, dt + u(X_h) e^{-\lambda h} \right\}; \tag{4.19}$$

see [24] for a proof which easily adapts to the current situation. In view of (4.4), (4.6) we know that there is a constant C such that $|X_t| \leqslant \|\bar{x}\| + Ct$ for $t \in (0, 1]$. For later simplicity, we observe that adding constants to φ and g we can assume that

$$\varphi(\bar{x}) = -u(\bar{x}), \qquad g(\bar{x}) = 0$$

and we do so hereafter. Then we have

$$u(X_h) + \varphi(X_h) + g(X_h) \geqslant u(\bar{x}) + \varphi(\bar{x}) + g(\bar{x}) = 0$$

and so, in view of (4.19) with $x = \bar{x}$, we see that

$$u(\bar{x}) \geqslant \inf_{\alpha_t} \left[\int_0^h f(X_t, \alpha_t) e^{-\lambda t}\, dt - \{\varphi(X_h) + g(X_h)\} e^{-\lambda h} \right]. \tag{4.20}$$

We then observe that we have

$$X_t = e^{-tA} \bar{x} + \int_0^t e^{-(t-s)A} b(X_s, \alpha_s)\, ds \tag{4.21}$$

and thus, in particular, we deduce

$$\|X_t - e^{-tA} \bar{x}\| \leqslant Ct \qquad \text{for} \quad t \in [0, 1] \tag{4.22}$$

for some positive constant C (independent of α_t). From (4.20), (4.22) and the continuity of f, φ, and g we deduce that

$$\lambda u(\bar{x}) + \sup_{\alpha_t} \left[\frac{1}{h} (\varphi(X_h) - \varphi(\bar{x})) + \frac{1}{h} g(X_h) - \frac{1}{h} \int_0^h f(\bar{x}, \alpha_t)\, dt \right] \geqslant -\varepsilon(h) \to 0$$

as $h \to 0^+$ since (4.22) implies that $X_t \to \bar{x}$ as $t \to 0^+$ uniformly in α_t. Because of (2.4) one has

$$\varphi(X_h) = \varphi(\bar{x}) + \int_0^h \langle \nabla\varphi(X_s), b(X_s, \alpha_s) \rangle\, ds$$
$$- \int_0^h \langle A^*\nabla\varphi(X_s), X_s \rangle\, ds. \tag{4.23}$$

For example, one can prove (4.23) by approximating A by its Yosida approximation $A_\varepsilon = A(I + \varepsilon A)^{-1}$, writing the corresponding formula in this case, and passing to the limit as $\varepsilon \downarrow 0$ while using that $A_\varepsilon^*\nabla\varphi(z) = (I + \varepsilon A^*)^{-1} A^*\nabla\varphi(z)$. Because of (2.4), (4.23) yields

$$\frac{1}{h} (\varphi(X_h) - \varphi(\bar{x})) + \langle A^*\nabla\varphi(\bar{x}), \bar{x} \rangle - \frac{1}{h} \int_0^h \langle \nabla\varphi(\bar{x}), b(\bar{x}, \alpha_t) \rangle\, dt \to 0$$

uniformly in α_t as $h \downarrow 0$. On the other hand, (4.21) and the monotonicity of A imply

$$\|X_t\|^2 \leqslant \|\bar{x}\|^2 + 2 \left\langle e^{-tA}\bar{x}, \int_0^t e^{-(t-s)A} b(X_s, \alpha_s)\, ds \right\rangle$$
$$+ \left\| \int_0^t e^{-(t-s)A} b(X_s, \alpha_s)\, ds \right\|^2.$$

Therefore, if $\bar{x} \neq 0$, we have an estimate

$$\|X_t\| \leqslant \|\bar{x}\| + \left\langle \frac{\bar{x}}{\|\bar{x}\|}, \int_0^t e^{-(t-s)A} b(\bar{x}, \alpha_s)\, ds \right\rangle + t\delta(t)$$

uniformly in α_t, where $\delta(t) \to 0$ as $t \downarrow 0$. Using this last estimate and (2.5) we deduce that if $\bar{x} \neq 0$ then

$$\frac{1}{h} g(X_h) \leqslant \left\langle \nabla g(\bar{x}), \frac{1}{h} \int_0^h e^{-(h-t)A} b(\bar{x}, \alpha_t)\, dt \right\rangle + \delta(h)$$
$$= \frac{1}{h} \int_0^h \langle e^{-(h-t)A^*} \nabla g(\bar{x}), b(\bar{x}, \alpha_t)\, dt \rangle + \delta_1(h)$$
$$\leqslant \left\langle \nabla g(\bar{x}), \frac{1}{h} \int_0^h b(\bar{x}, \alpha_t)\, dt \right\rangle + \delta'(h),$$

where $\delta'(h) \to 0$ as $h \to 0^+$. If $\bar{x} = 0$, we just observe that $\bar{x}$ is a minimum of $u + \varphi$ and eliminate g in this way. Collecting these estimates we finally deduce

$$\lambda u(\bar{x}) + \sup_{\alpha_t} \left[\frac{1}{h} \int_0^h (\langle b(\bar{x}, \alpha_t), \nabla\varphi(\bar{x}) + \nabla g(\bar{x})\rangle - f(\bar{x}, \alpha_t))\, dt \right]$$
$$+ \langle -A^* \nabla\varphi(\bar{x}), \bar{x}\rangle \geqslant \kappa(h) \to 0$$

as $h \downarrow 0$. Since the quantity in brackets becomes

$$\sup_{\alpha} \left[\langle b(\bar{x}, \alpha), \nabla\varphi(\bar{x}) + \nabla g(\bar{x})\rangle - f(\bar{x}, \alpha) \right]$$

we are done.

We next study the B-continuity of u and the uniqueness of u as a viscosity solution of (4.14), beginning with the strong B case.

THEOREM 4.3. *Assume* (3.5) *and* (4.2).

(i) *If* (4.5), (4.11), *and* (4.12) *also hold, then* $u \in BUC(H_{-1})$ *and thus* u *is B-continuous.*

(ii) *If* (4.5), (4.9), *and* (4.13) (*respectively,* (4.3), (4.8), *and* (4.13)) *also hold, then* u *is B-continuous and* u *is the unique B-continuous viscosity solution of* (4.14) *satisfying* (4.15) *for some* $\kappa < \lambda/C_2$ (*respectively,* (4.16) *for some* $\kappa < \lambda/C_0$).

Remark 4.4. Note that the uniqueness statements in (ii) are not consequences of the results stated and proved in Section III. The uniqueness assertion under (i) is covered by Section III since in that case one may work with $BUC(H)$ solutions.

However, even though the uniqueness is not a consequence of Section III, we will not prove it here since it follows from the techniques introduced here and ones found in [13] for the finite dimensional case. Note that the auxiliary functions used in [13] are nondecreasing and radial and thus can be used in the current setting in view of the definitions of viscosity solutions.

We prove the B-continuity statements. To this end, we first prove a B-version of (4.17). Indeed, with the notations of the proof of Theorem 4.1 we observe that

$$\frac{1}{2} \frac{d}{dt} \langle B(X_t^2 - X_t^1), X_t^2 - X_t^1\rangle + \langle A^* B(X_t^2 - X_t^1), X_t^2 - X_t^1\rangle$$
$$= \langle B(X_t^1 - X_t^2), b(X_t^1, \alpha_t) - b(X_t^2, \alpha_t)\rangle.$$

Using (3.5), (4.2) and the Cauchy–Schwarz inequality, we deduce easily that there exists a positive constant K_0 such that for all $x^1, x^2 \in H$, α_t

$$\frac{d}{dt}\langle B(X_t^2 - X_t^1), X_t^2 - X_t^1\rangle + \|X_t^2 - X_t^1\|^2$$
$$\leqslant 2K_0\langle B(X_t^2 - X_t^1), X_t^2 - X_t^1\rangle;$$

and hence

$$\langle B(X_t^1 - X_t^2), X_t^1 - X_t^2\rangle + \int_0^t e^{2K_0(t-s)}\|X_s^1 - X_s^1\|^2\, ds$$
$$\leqslant e^{2K_0 t}\langle B(x^1 - x^2), x^1 - x^2\rangle^{1/2}.$$

In particular,

$$\int_0^t \|X_s^1 - X_s^1\|\, ds \leqslant e^{K_0 t}\langle B(x^1 - x^2), x^1 - x^2\rangle^{1/2}(2K_0)^{-1/2}. \tag{4.24}$$

Then, if (4.5), (4.11), and (4.12) hold, we deduce for all $T \in (0, \infty)$

$$|u(x^1) - u(x^2)| \leqslant \int_0^T m(\|X_t^1 - X_t^2\|)e^{-\lambda t}\, dt + 2Ce^{-\lambda T}.$$

Next we recall that for all $\varepsilon > 0$ there exists $C_\varepsilon \geqslant 0$ such that

$$m(r) \leqslant \varepsilon + C_e r \qquad \text{for} \quad r \geqslant 0.$$

Hence we obtain, using (4.24), that

$$|u(x^1) - u(x^2)| \leqslant \frac{\varepsilon}{\lambda} + C_\varepsilon(2K_0)^{-1/2}\left(\int_0^T e^{(K_0 - \lambda)t}\, dt\right)$$
$$\times\langle B(x^1 - x^2), x^1 - x^2\rangle^{1/2} + 2Ce^{-\lambda T};$$

since $T > 0$ is arbitrary, this guarantees that $u \in BUC(H_{-1})$.

The other cases are treated similarly: indeed, if $x^1, x^2 \in B_R$ one shows in an analogous way that

$$|u(x^1) - u(x^2)| \leqslant \frac{\varepsilon}{\lambda} + C_\varepsilon\langle B(x^1 - x^2, x^1 - x^2\rangle^{1/2} + C_R e^{-\nu T},$$

where C_ε depends only on R, T, C_R depends only on R, and $\nu > 0$, is independent of ε, T, x^1, x^2. The B-continuity of u is established.

We proceed with the weak B case for which we need to make some further assumptions on b and f; namely

$$\langle b(x, \alpha) - b(y, \alpha), B(x-y)\rangle \leqslant C_1 \langle B(x-y), x-y\rangle \quad \text{for} \quad x, y \in H, \quad \alpha \in \mathscr{A} \tag{4.25}$$

or

$$\langle b(x, \alpha) - b(y, \alpha), B(x-y)\rangle \leqslant C_1^R \langle B(x-y), x-y\rangle \quad \text{for} \quad x, y \in B_R, \quad \alpha \in \mathscr{A}, \quad R>0 \tag{4.25$'$}$$

for some nonnegative constants C_1, C_1^R and

$$|f(x, \alpha) - f(y, \alpha)| \leqslant m(\langle B(x-y), x-y\rangle^{1/2}) \qquad \text{for} \quad x, y \in H, \quad \alpha \in \mathscr{A} \tag{4.26}$$

or

$$|f(x, \alpha) - f(y, \alpha)| \leqslant m_R(\langle B(x-y), x-y\rangle^{1/2}) \quad \text{for} \quad x, y \in B_R, \quad \alpha \in \mathscr{A}, \quad R>0, \tag{4.26$'$}$$

where $m(0^+) = 0$. Clearly (4.25) yields

$$\langle B(X_t^1 - X_t^2), X_t^1 - X_t^2\rangle \leqslant e^{C_1 t} \langle B(x^1 - x^2), x^1 - x^2\rangle$$

and therefore the B-continuity follows as in the proof of Theorems 4.1 and 4.3. We will not prove the uniqueness statements for the same reasons as given above and thus we will not give the proof of the following theorem:

THEOREM 4.5. *We assume* (4.2), $(3.5)_w$, *and* (4.10).

(i) *If also* (4.5), (4.11), (4.12), (4.25), *and* (4.26) *hold, then* $u \in BUC(H_{-1})$ *and thus* u *is B-continuous.*

(ii) *If also* (4.5), (4.9), (4.13), (4.25)$'$, *and* (4.26)$'$ *hold* (*respectively,* (4.3), (4.8), (4.13)), *then* u is *B-continuous and* u *is the unique B-continuous viscosity solution of* (4.14) *satisfying* (4.15) *for some* $\kappa < \lambda/C_2$ (*respectively,* (4.16) *for some* $\kappa < \lambda/C_0$).

Remark 4.6. It is worth observing that (4.25), (4.26) are precisely the assumptions needed to insure that

$$F(x, p) = \sup_{\alpha \in \mathscr{A}} [-\langle b(x, \alpha), p\rangle - f(x, \alpha)]$$

satisfies (3.9).

V. Differential Games and Existence for Stationary Problems

Let us begin with:

Proof of Existence for Theorem 3.1 (*Strong B Case*). Our strategy is to approximate F in a succession of stages, using various approximations which are consistent with the assumptions (3.1), (3.3)—that is, (3.1), (3.3) hold uniformly (with the same ω, σ) for the type of truncation used at each stage (but not uniformly over all the truncations which will appear). Existence for the crudest approximations of F under consideration will be proved by considering the value function of some differential game and using the sort of verification arguments given in the preceding section. Hence, in this sense, the existence program follows the one introduced in [12, Part II] of this series. Existence for successively more accurate approximations will then be proved by constructive limiting arguments. The tool we need to pass to the limit in the preliminary stages is:

Lemma 5.1. *Let* $G_n: H\times H \to \mathbb{R}$ *for* $n=1, 2, \ldots$ *satisfy* (3.1), (3.3) *uniformly in* n *and* $G_n(\cdot, 0)$ *be bounded uniformly in* n. *Let there be corresponding bounded B-continuous solutions* u_n *of*

$$u_n + \langle Ax, \nabla u_n\rangle + G_n(x, \nabla u_n) = 0 \qquad \text{in } H. \tag{5.1$_n$}$$

Suppose, moreover, $G_n(x, p) \to G(x, p)$ *uniformly on bounded subsets of* $H\times H$. *Then there exists* $u \in BUC(H_{-1})$ *such that* $u_n \to u$ *uniformly on bounded subsets of* H *and* u *is a solution of*

$$u + \langle Ax, \nabla u\rangle + G(x, \nabla u) = 0 \qquad \text{in } H. \tag{5.2}$$

Proof. Indeed, by the comparison assertions of Theorem 3.1 and the uniform bound on $G_n(\cdot, 0)$, the u_n are uniformly bounded in H. Moreover, u_n is uniformly continuous on H_{-1} uniformly in n. In order to prove this last assertion, we recall from the proof of Theorem 4.3 in [12, Part IV] that for all $\delta > 0$ it is possible to find $\varphi_\delta \in C^1([0, 1])$ such that $\varphi_\delta(0) = \delta$, $\varphi'_\delta(0) = 0$, $\varphi_\delta(1) \geqslant M \geqslant 2 \sup_{n,H} |u_n|$ and $\psi_\delta(x, y) = \varphi_\delta(\|x-y\|_{-1})$ is a supersolution of the "doubled" equation

$$\psi_\delta(x, y) + \langle Ax, \nabla_x \psi_\delta\rangle + \langle Ay, \nabla_y \psi_\delta\rangle + G_n(x, \nabla_x \psi_\delta) - G_n(y, -\nabla_y \psi_\delta) \geqslant 0$$

on the "strip" $\Delta = \{(x, y) \in H\times H : \|x-y\|_{-1} < 1\}$. Then the comparison proof of Theorem 3.1 adapted to the doubled equations yields

$$|u_n(x) - u_n(y)| \leqslant \psi_\delta(\|x-y\|_{-1}) \qquad \text{for} \quad (x, y) \in \Delta, \quad \delta > 0$$

and our claim is proved. Note that the success of this argument depended on choosing ψ_δ to be independent of n, which was possible because of the assumption that G_n satisfies (3.1) uniformly in n.

Finally, the convergence of the u_n also follows from the comparison part of Theorem 3.1; indeed, observe that in the proof as naturally adapted to estimate the difference $u_n - u_m$ of solutions of $(5.1)_n$ and $(5.1)_m$, the various parameters can be chosen uniformly in n, m. Moreover, for fixed values of the parameters the points in H entering in the nonlinearities are uniformly bounded in view of the above uniform modulus of continuity. Since G_n converges uniformly to G on balls, we deduce in this way that for all $x, y \in H$, $n, m \geqslant 1$

$$u_n(x) - u_m(y) - \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda(\mu(x) + \mu(y))$$
$$\leqslant \gamma(\varepsilon, \lambda) + \omega_{\varepsilon,\lambda}(n, m), \tag{5.3}$$

where $\omega_{\varepsilon,\lambda}(n, m) \to 0$ as $n, m \to \infty$ for fixed $\varepsilon, \lambda > 0$ and $\gamma(\varepsilon, \lambda) \to 0$ in the iterated limit $\lambda \to 0$ and then $\varepsilon \to 0$. Clearly, this implies that u_n converges uniformly bounded sets in H to some $u \in BUC(H_{-1})$. Proposition 2.7 then provides the information that u is a solution of (5.2) with the required properties.

We continue with the proof of Theorem 3.1, and argue that we may assume that F is bounded in addition to the other hypotheses, provided that we eventually establish existence in the bounded case. Indeed, existence in the unbounded case follows from existence in the bounded case by a variant of the argument used in Lemma 5.1. Setting $F_n = (F \wedge n) \vee (-n)$ for $n \geqslant 1$, it is a straightforward exercise to check that (3.1), (3.3), and (3.6) hold for F_n in place of F if they hold for F. Noting that F_n is bounded, we assume the existence of a solution of $u_n \in BUC(H_{-1})$ of

$$u_n + \langle Ax, \nabla u_n\rangle + F_n(x, \nabla u_n) = 0.$$

Then, since (3.6) holds with F_n in place of F for all n, we deduce from the comparison result in Theorem 3.1 that

$$u_n(x) - u_n(y) \leqslant w(x, y), \qquad \text{for all} \quad x, y \in H, \quad n \geqslant 1.$$

Indeed, w is a supersolution of the "doubled" equation and is also B-lower semicontinuous; this is enough to apply the comparison proof. The comparison proof now yields a uniform modulus of continuity in H_{-1} and we may conclude as before.

We may therefore assume that F is bounded. While we will construct approximations of F in several layers, in view of Lemma 5.1 we only have to check the uniform validity of (3.1), (3.3) at each layer of approximation (as well as remark that the bound on F is preserved at each stage), even if in order to do so we have to use assumptions on the Hamiltonian which are only valid at the preceding layer of approximation.

The next approximation process is

$$F_n(x, p) = \chi_n(\|x\|) F(x, p),$$

where the χ_n are cut-off functions satisfying $\chi_n(r) = \chi(r/n)$, where $\chi \in \mathscr{D}(\mathbf{R})$, $0 \leqslant \chi \leqslant 1$, $\chi(r) \equiv 1$ for $|r| \leqslant 1$. It is easy to see that the F_n satisfy (3.1) and (3.3) uniformly (with choices of ω, σ depending on the bound on F) since F satisfies (3.1), (3.3) and the F_n are uniformly bounded. Thus we may assume that

$$F(x, p) \text{ has bounded support in } x \text{ uniformly in } p. \tag{5.4}$$

We continue, now assuming that (5.4) holds as well as that F is bounded. This time we set

$$F_n(x, p) = \chi_n(\|p\|) F(x, p);$$

once again F_n satisfies (3.1), (3.3) uniformly in n (with the parameter functions depending on (5.4) and the bound on F). In view of the assumption that (the original) F was uniformly continuous on bounded sets, we have now that

$$F_n \in BUC(H) \text{ and the support of } F_n \text{ is bounded.} \tag{5.5}$$

Finally, we assume that F satisfies (5.5) and put

$$F_n(x, p) = \inf_{y, q \in H} \{F(y, q) + n \|x - y\| + n \|p - q\|\}.$$

Since F is bounded and uniformly continuous, F_n enjoys the same bound and modulus of continuity as F and therefore trivially satisfies (3.1), (3.3) uniformly in n. Moreover, F_n is Lipschitz continuous and $F_n \to F$ uniformly on bounded sets. Thus we need only establish the existence of a $BUC(H_{-1})$ solution of (S) in the case in which F is bounded and Lipschitz continuous.

This is the point where we introduce some simple differential games as in [12, Part II]: letting L be a Lipschitz constant for F in (x, p), we observe that if $x \in H$, $p \in B_L$, then

$$\begin{aligned} F(x, p) &= \inf_{\|q\| \leqslant L} (F(x, q) + L \|p - q\|) \\ &= \inf_{\|q\| \leqslant L} \sup_{\|z\| \leqslant L} \{-(p, z) + f(x, z, q)\}, \end{aligned} \tag{5.6}$$

where

$$f(x, z, q) = (q, z) + F(x, q).$$

Letting $\tilde{F}$ be defined on $H \times H$ by the right-hand side of (5.6), we note that $\tilde{F}$ still has L as a Lipschitz constant and $\tilde{F} = F$ on $H \times B_L$. We will show the existence of a solution $u \in BUC(H_{-1})$ of (S) with F replaced by $\tilde{F}$ which further satisfies

$$|u(x) - u(y) \leqslant L \, \|x - y\|; \tag{5.7}$$

it will follow in a standard way that u is also a solution of (S).

The solution, whose existence is being proved, may be given as in [12, Part II]. Let

$$\mathscr{A} = \{\text{strongly measurable } q\colon [0, \infty) \to B_L\}$$

and

$$\mathscr{B} = \{\text{nonanticipating maps } Z\colon \mathscr{A} \to \mathscr{A}\},$$

where nonanticipating means that if $q_1 = q_2$ a.e. on an interval $[0, T]$, then $Z(q_1) = Z(q_2)$ a.e. on $[0, T]$. For all $q \in \mathscr{A}$ and $Z \in \mathscr{B}$, and initial states $x \in X$, we define the state X_t for $t \geqslant 0$ by

$$\frac{d}{dt} X_t + AX_t = Z(q_t), \qquad X_0 = x \in H, \tag{5.8}$$

the "cost functional" by

$$J(x, Z, q) = \int_0^\infty f(X_t, Z(q)_t, q_t) e^{-t} \, dt, \tag{5.9}$$

and, finally the "value" function is given by

$$u(x) = \inf_{Z \in \mathscr{B}} \sup_{q \in \mathscr{A}} J(x, Z, q). \tag{5.10}$$

Since $X_t = e^{-tA} x + \int_0^t e^{-(t-s)A} Z(q_s) \, ds$ and f is Lipschitz in x with constant L, it is evident that u also has L as a Lipschitz constant and (5.7) holds. One shows that $u \in BUC(H_{-1})$ as in the preceding section. A straightforward combination of arguments like those in the preceding section and in [15; 30; 12, Part II] shows that u is a solution of (S).

Remark 5.2. If $F(\cdot, 0)$ is bounded, just as in [12, Part IV] one sees that both the existence and uniqueness result are valid if we replace (3.1) by

$$\left.\begin{aligned} &F(x, B^{1/2}p) - F(x, B^{1/2}p + \lambda \nabla \mu(x)) \leqslant \sigma_R(\lambda), \\ &F(x, B^{1/2}p - \lambda \nabla \mu(x)) - F(x, B^{1/2}p) \leqslant \sigma_R(\lambda), \end{aligned}\right. \tag{5.11}$$

for all $\lambda \geqslant 0$, $x \in H$, $p \in B_R$, and $R > 0$.

We turn to the proof of existence in Theorem 3.1 in the weak B case.

Proof of Existence for Theorem 3.1 (*Weak B Case*). The proof is divided into two parts. The first part consists in showing by a simple "viscosity" method perturbation argument that there exists a solution of (S) when the definition of solution is slightly modified as follows: we will not ask that the test functions φ in the definition satisfy (2.4), but instead φ should satisfy

$$\left.\begin{aligned} &\varphi \text{ is weakly sequentially lower semicontinuous on } H \text{ and} \\ &\nabla\varphi,\ B^{-1}\nabla\varphi \text{ are continuous.} \end{aligned}\right\} \tag*{(2.4)$'$}$$

We will refer to solutions of this sort as solutions in the sense of Definition 2.1$'$. Note that (2.4) and (2.4)$'$ are equivalent in the strong B case. However, while (2.4)$'$ implies (2.4), the converse is not true in general. Nonetheless, the second part of the existence proof will show that solutions in the sense of Definition 2.1$'$ are solutions in the sense of Definition 2.1 when the weak B condition holds.

We begin by noting that the comparison statements of Theorem 3.1 remain valid for solutions in the sense of Definition 2.1$'$ since the test functions used in the proofs clearly satisfy (2.4)$'$. We introduce perturbations A_n of A by letting the adjoint A_n^* of A_n be given by the formula

$$A_n^* = A^* + \frac{1}{n} B^{-1}. \tag{5.12}$$

This is possible since our assumptions imply that $A^* + (1/n)B^{-1}$ is maximal monotone (note, in particular, that $D(B^{-1}) = R(B) \subset D(A^*)$ by B). In view of the weak B assumption and (5.12) we have

$$\langle A_n^* Bx + C_0 Bx, x \rangle \geqslant \frac{1}{n} \|x\|^2 \tag{5.13}$$

so A_n satisfies the strong B condition (3.5) up to the irrelevant factor $1/n$. Hence, by the strong B case, there exists a solution $u_n \in UC(H_{-1})$ of

$$u_n + \langle A_n x, \nabla u_n \rangle + F(x, \nabla u_n) = 0. \tag{5.14}$$

The notion of solution is that of Definition 2.1 with A_n in place of A and in view of the definition of A_n this involves only test functions satisfying

(2.4)'. Moreover, as in [12, Part IV], we have a uniform modulus of continuity

$$|u_n(x) - u_n(y)| \leqslant m(\|x-y\|_{-1}). \tag{5.15}$$

We claim that the u_n converge to a limit u. Indeed, the uniqueness and comparison proofs adapted to compare u_n and u_m and using (5.15) yield an estimate

$$\begin{aligned} &u_n(x) - u_m(y) + \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda(\mu(x)+\mu(y)) \\ &\leqslant \mu_\varepsilon(\lambda) + \sigma(\varepsilon) + \left(\frac{1}{n}+\frac{1}{m}\right) M_{\varepsilon,\lambda}, \end{aligned}$$

where $M_{\varepsilon,\lambda} \geqslant 0$ is independent of n, m, $\mu_\varepsilon(\lambda) \to 0$ as $\lambda \downarrow 0$ for fixed $\varepsilon > 0$ and $\sigma(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Thus we see tht u_n converges uniformly on bounded subsets of H to some $u \in BUC(H_{-1})$. By the proof of the stability results in Section 2, we see that u is a solution of (S) in the sense of Definition 2.1'.

We now turn to the second part of the proof, which consists of establishing the following:

PROPOSITION 5.3. *Let the assumptions of Theorem 3.7 hold and let $u \in BUC(H_{-1})$ be a solution of (S) in the sense of Definition 2.1'. Then u is a solution of (S).*

Proof. The first step consists of a regularization of u (as in [27]) indeed, for $\varepsilon, \lambda \in (0, 1)$, we introduce

$$\tilde{u}(x) = \sup_{y \in H} \left\{ u(y) - \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda\mu(y)\right\}. \tag{5.16}$$

In view of (5.15), which holds for u in place of u_n, one has

$$\begin{aligned} u(x) - \lambda\mu(x) \leqslant \tilde{u}(x) &\leqslant u(x) + \sup_{y \in H}\left\{ m(\|x-y\|_{-1}) - \frac{1}{2\varepsilon}\|x-y\|_{-1}^2\right\} \\ &= u(x) + \delta(\varepsilon), \end{aligned} \tag{5.17}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. We also observe that

$$\tilde{u}(x) = \sup_{\|y\| \leqslant K} \left\{ u(y) - \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda\mu(y)\right\} \qquad \text{for} \quad x \in B_R, \tag{5.18}$$

where K depends only on R and λ,

$$-C(1+\lambda\,\|x\|) \leqslant \tilde{u}(x) \leqslant C \tag{5.19}$$

holds on H for some C,

$$|\tilde{u}(x)-\tilde{u}(y)| \leqslant M\,\|x-y\|_{-2} \qquad \text{for} \quad x, y \in B_R \tag{5.20}$$

for some M depending only on R, ε, and λ.

We next claim that $\tilde{u}$ satisfies, in the sense of Definition 2.1′, an inequality of the form

$$\tilde{u} + \langle Ax, \nabla\tilde{u}\rangle + F(x, \nabla\tilde{u}) \leqslant r(\|x\|, \varepsilon, \lambda) \tag{5.21}$$

on H, where $r(R, \varepsilon, \lambda)$ is continuous and nondecreasing with respect to R for all ε, $\lambda > 0$ and for $R > 0$ we have $r(R, \varepsilon, \lambda) \to 0$ in the iterated limit $\lambda \downarrow 0$ and then $\varepsilon \downarrow 0$.

To this end, let x_0 be a global strict maximum of $\tilde{u} - \varphi - g$, where φ is weakly lower-semicontinuous, continuously differentiable and $\nabla\varphi$, $B^{-1}\nabla\varphi \in C(H)$ and we assume that for $r \geqslant \delta > 0$ there is a $v_\delta > 0$ such that $g'(r) \geqslant v_\delta$. Without loss of generality, we may assume that φ is bounded from above and $g(r)/r \to \infty$ as $r \to \infty$.

By arguments similar to those introduced in Section 2, we may find an $R \in (1, \infty)$ such that for all $\alpha > 0$ there exists $p, q \in H$ with $\|p\|, \|q\| \leqslant 1$ and

$$\begin{aligned} u(y) &- \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda\mu(y) - \varphi(x) - g(x) \\ &\leqslant \tilde{u}(x_0) - \lambda\mu(x_0) - \varphi(x_0) - g(x_0) - 1 \qquad \text{for} \quad \|x\| + \|y\| \geqslant R-1, \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} \Psi(x, y) = u(y) &- \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \lambda\mu(y) - \varphi(x) \\ &- g(x) - \alpha\langle Bq, y\rangle - \alpha\langle Bq, x\rangle \end{aligned} \tag{5.23}$$

admits a local maximum on $B_R \times B_R$ at $(\bar{x}, \bar{y})$ and $\|\bar{x}\|, \|\bar{y}\| \leqslant R-1$. Since

$$\Psi(\bar{x}, \bar{y}) \geqslant \tilde{u}(x_0) - \varphi(x_0) - g(x_0) - C\alpha$$

and

$$\Psi(\bar{x}, \bar{y}) \leqslant \tilde{u}(\bar{x}) - \varphi(\bar{x}) - g(\bar{x}) + C\alpha$$

for some C independent of $\alpha > 0$, we deduce easily that $\bar{x}$ converges in H to x_0 as $\alpha \downarrow 0$.

The proof of (3.17) then yields

$$\|B(\bar{x}-\bar{y})\| \leqslant C_\varepsilon \qquad \text{for some } C_\varepsilon \text{ independent of } \alpha, \lambda \in (0,1). \tag{5.24}$$

Finally, remarking that we have

$$u(\bar{y}) - \frac{1}{2\varepsilon}\langle B(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle - \lambda\mu(\bar{y}) \geqslant \tilde{u}(\bar{x}) - C\alpha$$

for some C independent of ε, λ, α, we deduce easily

$$\frac{1}{2\varepsilon}\langle B(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle \leqslant m(\varepsilon) + 2\lambda\mu(\bar{x}) + 2C\alpha, \tag{5.25}$$

where $m(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. We now apply the proof of uniqueness in Theorem 3.7 to obtain successively

$$u(\bar{y}) + \left\langle \bar{y}, \alpha A^*Bp + \frac{1}{\varepsilon} A^*B(\bar{y}-\bar{x})\right\rangle$$
$$+ F\left(\bar{y}, \alpha Bp + \frac{1}{\varepsilon} B(\bar{y}-\bar{x}) + \lambda\nabla\mu(\bar{y})\right) \leqslant 0,$$
$$\tilde{u}(\bar{x}) + \left\langle \bar{x}, \frac{1}{\varepsilon} A^*B(\bar{y}-\bar{x})\right\rangle + F\left(\bar{x}, \frac{1}{\varepsilon} B(\bar{y}-\bar{x})\right)$$
$$\leqslant \gamma_{\varepsilon,\lambda}(\alpha) + \gamma_\varepsilon(\lambda) + \omega\left(\|\bar{x}-\bar{y}\|_{-1} + \frac{1}{\varepsilon}\|\bar{x}-\bar{y}\|_{-1}^2\right)$$
$$+ \frac{C_0}{\varepsilon}\langle B(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle'\, \tilde{u}(\bar{x}),$$
$$+ \tilde{u}(\bar{x}) + \langle \bar{x}, A^*\nabla\varphi(\bar{x})\rangle + F(\bar{x}, \nabla\varphi(\bar{x}) + \nabla g(\bar{x}))$$
$$\leqslant \gamma'_{\varepsilon,\lambda}(\alpha) + \gamma_\varepsilon^{R_0}(\lambda) + \gamma(\varepsilon),$$

where we have used (5.25), the equation

$$\frac{1}{\varepsilon} B(\bar{x}-\bar{y}) = \nabla\varphi(\bar{x}) + \nabla g(\bar{x}) + \alpha Bq$$

which, in particular, shows that $\nabla g(\bar{x}) \in D(B^{-1})$ and thus $\langle A^*\bar{x}, \nabla g(\bar{x})\rangle \geqslant 0$, and where $R_0 = \|x_0\|$, $\gamma(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, $\gamma_{\varepsilon,\lambda}(\alpha)$, $\gamma'_{\varepsilon,\lambda}(\alpha) \to 0$ as $\alpha \downarrow 0$ for fixed ε, $\lambda > 0$, $\gamma_\varepsilon(\lambda)$, $\gamma_\varepsilon^{R_0}(\lambda) \to 0$ as $\lambda \downarrow 0$ for fixed $\varepsilon > 0$. Letting $\alpha \downarrow 0$ we recover

$$\tilde{u}(x_0) + \langle x_0, A^*\nabla\varphi(x_0)\rangle + F(x_0, \nabla\varphi(x_0) + \nabla g(x_0)) \leqslant \gamma_\varepsilon^{R_0}(\lambda) + \gamma(\varepsilon)$$

proving our claim.

We just have to show that (5.21) holds in the sense of Definition 2.1 in order to conclude the proof of Proposition 5.3 in view of (5.17) (which shows that $\tilde{u} \to u$ uniformly on bounded subsets of H as $\varepsilon, \lambda \downarrow 0$), (5.21), the fact that for each R there are sequences $\lambda_n, \varepsilon_n \downarrow 0$ such that $r(R, \varepsilon_n, \lambda_n) \to 0$, and the stability results of Section 2 (in fact, this concludes the proof of the subsolution part and the supersolution part is proven in a similar manner). This remaining fact is in fact much more general and is a direct consequence of (5.20) and the following:

PROPOSITION 5.4. *Let $u \in C(H)$ be a subsolution (respectively, supersolution) of (S) in the sense of Definition 2.1′ such that u satisfies (5.20). Let $\varphi \in C^1(H)$ and let x_0 be a local maximum (respectively, minimum) point of $u - \varphi$. Then $\nabla\varphi(x_0) \in D(B^{-1}) \subset D(A^*)$ and we have*

$$u(x_0) + \langle x_0, A^*\nabla\varphi(x_0)\rangle + F(x_0, \nabla\varphi(x_0)) \leqslant 0 \tag{5.26}$$

(*respectively,*

$$u(x_0) + \langle x_0, A^*\nabla\varphi(x_0)\rangle + F(x_0, \nabla\varphi(x_0)) \geqslant 0.) \tag{5.27}$$

Proof. Let x_0 be a local maximum point of $u - \varphi$, where $\varphi \in C^1(H)$. By standard considerations (similar to the proof of (3.17)), (5.20) implies that $p_0 = \nabla\varphi(x_0) \in D(B^{-1}) \subset D(A^*)$. We consider next the case $\varphi \in C^{1,1}(H)$, i.e., the case in which $\nabla\varphi$ is Lipschitz continuous on H. Therefore we can find $\delta > 0$, $C_0 \geqslant 0$ such that u is bounded on $B(x_0, \delta)$ and

$$\left.\begin{aligned} &(u - \tilde{\varphi})(x) \leqslant (u - \tilde{\varphi})(x_0) - \|x - x_0\|^2 \qquad \text{for} \quad \|x - x_0\| \leqslant \delta, \\ &\quad \text{where} \quad \tilde{\varphi}(x) = \varphi(x_0) + \langle p_0, x - x_0\rangle + \tfrac{1}{2}C_0 \|x - x_0\|^2. \end{aligned}\right\} \tag{5.28}$$

We then choose a sequence $\{x_0^n\} \subset D(B^{-1})$ such that $x_0^n \to x_0$ in H. Next, by the (now familiar perturbation argument, one can find for all $\alpha > 0$ points $p \in B_1$, $\bar{x} \in B(x_0, \delta)$ such that $\bar{x}$ is a maximum of

$$u(x) - \varphi(x_0) - \langle p_0, x - x_0\rangle - \frac{C_0}{2}\|x - x_0^n\|^2 - \alpha\langle Bp, x\rangle$$

and for α small enough and n, large, $\|\bar{x} - x_0\| < \delta$. In addition, because of (5.28), $\bar{x} \to x_0$ as $\alpha \downarrow 0$, $n \to \infty$. Then, using (5.20), we deduce that $p_0 + C_0(\bar{x} - x_0^n) + \alpha Bp$ is bounded in H_2—or, in other words, $\bar{x} \in D(B^{-1}) \subset D(A^*)$ and

$$\|B^{-1}(\bar{x} - x_0^n)\| \leqslant C \qquad \text{for} \quad n \geqslant 0, \quad \alpha \in (0, 1). \tag{5.29}$$

Next, we remark that we may write

$$\varphi(x_0) + \langle p_0, x - x_0 \rangle + \frac{C_0}{2} \|x - x_0^n\|^2$$

$$= \left[\varphi(x_0) + \langle p_0, x - x_0 \rangle + C_0 \langle x, \bar{x} - x_0^n \rangle \right.$$

$$\left. + \frac{C_0}{2} (\|x_0^n\|^2 - \|\bar{x}\|^2) + \alpha \langle Bp, x \rangle \right] + \frac{C_0}{2} \|x - \bar{x}\|^2$$

and the quantity between the brackets is clearly weakly continuous and its gradient is constant and equal to $p_0 + C_0(\bar{x} - x_0^n) \in D(B^{-1}) \subset D(A^*)$. Next, we observe that $\bar{x} \in D(A^*)$ and that Remark 2.6 can still be applied for solutions in the sense of Definition 2.1′ with exactly the same proof (taken from [12, Part IV, Appendix] in fact). This is why we can deduce that

$$\begin{aligned} u(\bar{x}) + \langle \bar{x}, A^* \{ p_0 + C_0(\bar{x} - x_0^n) + \alpha Bp \} \rangle & \\ + F(\bar{x}, p_0 + C_0(\bar{x} - x_0^n) + \alpha Bp) \leqslant 0 & \qquad (5.30) \end{aligned}$$

and we conclude letting α go to 0 and n go to infinity recalling that $\bar{x}$ and x_0^n converge to x_0 in H, $p \in B_1$ and, because of (5.29), $A^*(\bar{x} - x_0^n)$ converges weakly in H to 0.

At this point, we have shown (5.26) when $\varphi \in C^{1,1}(H)$. Next we treat the general case. We remark that, without loss of generality, we may assume there exists $\delta > 0$, Φ increasing and C^1 on $[0, \delta]$ such that $\Phi(0) = \Phi'(0) = 0$ and

$$\begin{aligned} &(u - \tilde{\varphi})(x) \leqslant (u - \tilde{\varphi})(x_0) - \Phi(\|x - x_0\|) \qquad \text{for} \quad \|x - x_0\| \leqslant \delta \\ &\text{where} \quad \tilde{\varphi}(x) = \varphi(x_0) + \langle p_0, x - x_0 \rangle + \Phi(\|x - x_0\|). \end{aligned} \qquad (5.31)$$

Constructions of this sort are standard in the viscosity theory (see [11], for instance). Next, we consider $\Phi_n \in C^2([0, \delta]))$ such that $\Phi_n(0) = 0$, $\Phi_n'(0) = 0$ and Φ_n converges to Φ in $C^1([0, \delta])$. Then, exactly as above, we find for n large and α small, $p \in B_1$ and $\bar{x} \in \{ \|x - x_0\| < \delta \}$ such that

$$\begin{aligned} &\bar{x} \text{ is a maximum of } u(x) - \varphi(x_0) - \langle p_0, x - x_0 \rangle - \Phi_n(\|x - x_0\|) \\ &\quad - \alpha \langle Bp, x \rangle \text{ over } B(x_0, \delta). \end{aligned}$$

Then, applying the fact established above, we obtain

$$\begin{aligned} u(\bar{x}) + \left\langle \bar{x}, A^* \left\{ p_0 + \Phi_n'(\|\bar{x} - x_0\|) \frac{\bar{x} - x_0}{\|\bar{x} - x_0\|} + \alpha Bp \right\} \right\rangle & \\ + F(\bar{x}, p_0 + C_0(\bar{x} - x_0^n) + \alpha Bp) \leqslant 0 & \qquad (5.32) \end{aligned}$$

and we conclude upon recalling that $\bar{x}$ converges to x_0 in H, sending $\alpha \downarrow 0$, $n \to \infty$, and recalling that because of (5.20)

$$\Phi_n'(\|\bar{x} - x_0\|) \, \|x_0 - \bar{x}\|^{-1} \, \|B^{-1}(\bar{x} - x_0)\| \leqslant C,$$

where C is independent of $n \geqslant 1$, $\alpha \in (0, 1)$. This completes the proof of Proposition 5.4.

VI. Time Dependent Problems and Other Extensions

We turn to the case of (E), beginning with the strong B case.

Theorem 6.1. *Let* (3.1), (3.3), *and* (3.5) *hold.*

Comparison. Let $u, v \in UC_s([0, T] \times H) \cap C_b([0, T] \times H)$ *be, respectively, a subsolution and a supersolution of* (E) *on* $(0, T] \times H$. *Assume that* u, v *are B-continuous on* $(0, T] \times H$ *and satisfy*

$$\lim_{t \downarrow 0} \left((u(t, x) - u(0, e^{-tA}x))^+ + (v(t, x) - v(0, e^{-tA}x))^-\right) = 0 \quad \text{uniformly on } H. \tag{6.1}$$

If either u *or* v *are bounded from above or* (3.4) *holds, then*

$$u(t, x) - v(t, x) \leqslant \sup_{y \in H} (u(0, y) - v(0, y)) \qquad \text{for} \quad (t, x) \in [0, T] \times H. \tag{6.2}$$

Existence. Let $\psi \in UC(H)$ *and assume that*

$$F(t, x, p) \text{ is bounded on } [0, T] \times H \times B_R \text{ for } R > 0. \tag{6.3}$$

If either ψ *is bounded or* (3.4) *holds, then* (E) *has a unique solution* $u \in UC_s([0, T] \times H) \cap C_b([0, T] \times H)$ *such that* $u \in UC_s([\delta, T] \times H_{-1})$ *for all* $\delta > 0$, (6.1) *holds with* $v \equiv u$ *and* $u(0, x) = \psi(x)$ *on* H. *Moreover,* u *is bounded on* $[0, T] \times H$ *if* ψ *is bounded and there exists a modulus* ρ *such that*

$$|u(t, x) - u(s, x)| \leqslant \rho(t - s) \qquad \text{for} \quad 0 \leqslant s \leqslant t \leqslant T, \quad x \in H. \tag{6.4}$$

Remark 6.2. Remarks 3.3–3.5 readily adapt to the current case.

Remark 6.3. It is possible to assume (6.1) only on balls (rather than all of H) provided that we assume $u, v \in UC_s([0, T] \times H_{-1})$ and the existence assertion holds without assuming (6.3) provided $\psi \in UC(H_{-1})$, in which case the solution satisfies (6.1) on balls and belongs to $UC_s([0, T] \times H_{-1})$.

Note also that if $\psi \equiv 0$ (for instance), we may replace (6.3) with the condition that $F(t, x, 0)$ is bounded on $[0, T] \times H$.

We turn to the weak B case.

THEOREM 6.41. *Let* (3.3), $(3.5)_w$, *and* (3.9) *hold.*

Comparison. Let $u, v \in UC_s([0, T] \times H_{-1}) \cap C_b([0, T] \times H)$ be, respectively, a subsolution and a supersolution of (E) *in $(0, T] \times H$ such that*

$$\lim_{t \downarrow 0} (u(t, x) - u(0, x))^+ + (v(t, x) - v(0, x))^- = 0 \qquad \textit{uniformly on } H.$$

If either u and v are bounded above or (3.4) *holds, then* (6.2) *holds.*

Existence. If $\psi \in UC(H_{-1})$, and either (3.4) *holds or ψ and $F(t, x, 0)$ are bounded, then* (E) *has a unique solution $u \in UC_s([0, T] \times H_{-1}) \cap C_b([0, T] \times H)$ such that $u(t, x) \to \psi(x)$ as $t \downarrow 0$ uniformly on bounded subsets of H. This solution is bounded if ψ and $F(t, x, 0)$ are bounded. Moreover, for $R > 0$, $u(t, x)$ is uniformly continuous in t uniformly for $x \in B_R$.*

Remark 6.5. The analogues of Remarks 3.8, 3.9 still hold.

We will not prove Theorems 6.1, 6.4 here as the proofs are a combination of the arguments of [12] and those given in Sections 3–5. In particular, the analogues of the results of Section 4 will be found in the next section.

Instead, we prefer to present some extensions to "unbounded situations" (as $\|x\| \to \infty$) which will be useful for applications to optimal control problems in the next section. To this end, we first "localize" the crucial uniqueness structure conditions (3.1) and (3.9) in the forms

$$\begin{aligned} &F(t, y, \lambda B(x-y)) - F(t, x, \lambda B(x-y)) \\ &\qquad \leqslant \omega(\|x-y\| \, (1 + \lambda \, \|x-y\|_{-1}), R) \end{aligned} \tag{6.5}$$

for all $x, y \in B_R$, $t \in [0, T]$, $\lambda \geqslant 0$, and

$$\begin{aligned} &F(t, y, \lambda B(x-y)) - F(t, x, \lambda B(x-y))) \\ &\qquad \leqslant \omega(\|x-y\|_{-1} \, (1 + \lambda \, \|x-y\|_{-1}), R) \end{aligned} \tag{6.6}$$

for all $x, y \in B_R$, $t \in [0, T]$, $\lambda \geqslant 0$, where $\omega(\sigma, R) \to 0$ as $\sigma \downarrow 0$, for all $R \geqslant 0$ and $\omega(\sigma, R)$ is increasing and continuous with respect to $\sigma \geqslant 0$ and $R \geqslant 0$. Recall also that we always assume that F is uniformly continuous on $[0, T] \times B_R \times B_R$ for all $R \geqslant 0$, and that A, B hold. The condition that we will use in order to "localize" the uniqueness proofs is taken from [13],

$$|F(t, x, p) - F(t, x, q)| \leqslant \omega(\|p-q\| \, (1 + \|x\|)) \tag{6.7}$$

for all $p, q \in H$, $x \in H$, $t \in [0, T]$, where ω is continuous increasing on $[0, \infty)$ and $\omega(0) = 0$.

Then, by a straightforward combination of the arguments of [13] and the preceding ones, we obtain the following two results which are valid, respectively, in the strong and the weak B cases.

THEOREM 6.6. *Let* (3.5), (6.5), *and* (6.7) *hold.*

Comparison. Let $u, v \in UC_s([0, T) \times B_R) \cap C_b([0, T] \times H)$ *for every* $R \geqslant 0$ *be B-continuous on* $(0, T] \times H$ *and, respectively, a subsolution and a supersolution of* (E) *on* $(0, T) \times H$ *which satisfy*

$$\lim_{t \downarrow 0} \left((u(t, x) - u(0, e^{-tA}x))^+ + (v(t, x) - v(0, e^{-tA}x))^-\right) = 0 \quad \text{uniformly for } x \in B_R \tag{6.8}$$

for $R \geqslant 0$. *Then* (6.2) *holds.*

Existence. Let $\psi \in UC(B_R)$, *for* $R \geqslant 0$. *Then* (E) *has a unique solution* $u \in UC_s([0, T) \times B_R) \cap C_b([0, T] \times H)$ *for* $R \geqslant 0$ *which is B-continuous on* $(0, T] \times H$, *satisfies* (6.8) *with* $v \equiv u$ *and* $u(0, x) = \psi(x)$ *on* H. *Moreover, for all* $R \geqslant 0$, (6.4) *restricted to* B_R *holds.*

THEOREM 6.7. *We assume* $(3.5)_w$, (6.6), *and* (6.7).

Comparison. Let u, v *be uniformly continuous* $x \in B_R$ *in the* H_{-1} *norm, uniformly in* $t \in [0, T]$, *for* $R \geqslant 0$ *and assume that* $u, v \in C_b([0, T] \times H)$ *are, respectively, a subsolution and a supersolution of* (E) *and satisfy*

$$\lim_{t \downarrow 0} \left((u(t, x) - u(0, x))^+ + (v(t, x) - v(0, x))^-\right) = 0 \quad \text{uniformly for } x \in B_R \tag{6.9}$$

for $R > 0$. *Then* (6.2) *holds.*

Existence. If $\psi \in C_b(H)$ *is uniformly continuous on* B_R *for the* H_{-1} *norm for all* $R \geqslant 0$, *then* (E) *has a unique solution* $u \in C_b([0, T] \times H)$ *which is uniformly continuous in* $x \in B_R$ *for the* H_{-1} *norm, uniformly in* $t \in [0, T]$ *for each* $R \geqslant 0$ *and satisfies* (6.9) *with* $v \equiv u$ *and* $u(0, x) = \psi(x)$ *in* H. *Moreover,* $u(t, x)$ *is uniformly continuous in* $(t, x) \in [0, T] \times B_R$ *for all* $R \geqslant 0$.

VII. FINITE HORIZON OPTIMAL CONTROL PROBLEMS

We begin with the analogues for time dependent problems of the results given in Section 4. We keep the notations $\mathscr{A}$ and α_t for the control space and admissible controls and define, given an initial state $x \in H$, and an initial instant $s \in [0, T]$, for some $T \in (0, \infty)$ which is the horizon of the control problems, the state of the system for $t \geqslant s$ by the solution of

$$\frac{dX_t}{dt} + AX_t = b(t, X_t, \alpha_t), \qquad X_s = x, \tag{7.1}$$

where b is continuous on $[0, T] \times H \times \mathscr{A}$ and satisfies

$$\|b(t, x, \alpha) - b(t, y, \alpha)\| \leqslant C_0 \|x - y\| \tag{7.2}$$

for $x, y \in H$, $\alpha \in \mathscr{A}$, $t \in [0, T]$ and

$$\|b(t, 0, \alpha)\| \leqslant C_1, \qquad \text{for} \quad t \in [0, T], \quad \alpha \in \mathscr{A} \tag{7.3}$$

for some $C_0 \geqslant 0$, $C_1 \geqslant 0$. Clearly, there exists a unique solution X_t of (7.1) in $C([s, T], H)$ which satisfies

$$\|X_t\| \leqslant e^{C_0(t-s)} \|x\| + C_1 C_0^{-1}(e^{C_0(t-s)} - 1) \tag{7.4}$$

for $t \in [s, T]$. Similarly,

$$\|X_t\| \leqslant \|x\| + C_2(t - s) \tag{7.5}$$

for $t \in [s, T]$ provided b satisfies

$$\|b(t, x, \alpha)\| \leqslant C_2 \qquad \text{for} \quad t \in [0, T], \quad x \in H, \quad \alpha \in \mathscr{A}. \tag{7.6}$$

Given a continuous function f on $[0, T] \times H \times \mathscr{A}$, which is bounded on $[0, T] \times B_R \times \mathscr{A}$ for all $R \geqslant 0$, one then defines a (cost) function(al) J which now depends on $x \in H$, $s \in [0, T]$, α_t by

$$J(s, x, \alpha_t) = \int_s^T f(t, X_t, \alpha_t)\, dt + g(X_t), \tag{7.7}$$

where $g \in C_b(H)$. We will use the following conditions on f, g:

$$\left.\begin{array}{l} g \in BUC(H) \text{ and } f(t, x, \alpha) \text{ is bounded, uniformly continuous in} \\ x \in H \text{ uniformly for } (t, \alpha) \in [0, T] \times \mathscr{A} \end{array}\right\} \tag{7.8}$$

or

$$|g(x) - g(y)| + |f(t, x, \alpha) - f(t, y, \alpha)| \leqslant \omega(\|x - y\|, R)$$
$$\text{for} \quad x, y \in B_R, \quad t \in [0, T], \quad \alpha \in \mathscr{A} \tag{7.9}$$

for some continuous function $\omega(s, r)$, increasing in both variables $s, r \geqslant 0$ and such that $\omega(0, r) = 0$ for all $r \geqslant 0$. This condition merely means that f

is uniformly continuous in $x \in B_R$, uniformly in $t \in [0, T]$, $\alpha \in \mathscr{A}$ and g is uniformly continuous in $x \in B_R$, for all $R \geqslant 0$.

Finally we introduce the value function

$$u(s, x) = \inf_{\alpha_t} \left\{ \int_s^T f(t, X_t, \alpha_t)\, dt + g(X_T) \right\}, \qquad s \in [0, T], \quad x \in H \tag{7.10}$$

for where the infimum is taken over all admissible controls α_t.

We then have the following analogues, stated without proofs, of the results obtained in Section 4.

THEOREM 7.1. *Let* (7.2), (7.3), (7.9), and (7.10) *hold. Then u is uniformly continuous in $x \in B_R$ uniformly in $s \in [0, T]$ and satisfies* (6.4) *restricted to arbitrary balls B_R. In particular, $u \in C_b([0, T] \times H)$. If* (7.6) *and* (7.8) *also hold, then $u \in UC_s([0, T] \times H)$, u is bounded and* (6.4) *holds. Finally, u is a viscosity solution of*

$$-\frac{\partial u}{\partial s} + \sup_{\alpha \in \mathscr{A}} \{ -\langle b(s, x, \alpha), \nabla_x u(s, x) \rangle - f(s, x, \alpha) \} + \langle Ax, \nabla_x u(s, x) \rangle = 0 \tag{7.11}$$

in $(0, T) \times H$ *and* $u(T, x) = g(x)$ *on* H.

THEOREM 7.2. *We assume* (3.5), (7.2), (7.3), (7.9), *and* (7.10). *Then u satisfies*

$$|u(s, x) - u(s, y)| \leqslant \omega_\varepsilon(\|x - y\|_{-1}) \qquad \text{for} \quad x, y \in B_{1/\varepsilon}, \quad s \in (0, T - \varepsilon) \tag{7.12}$$

for some modulus ω_ε, where ε is arbitrary in $(0, T)$. If (7.6) *and* (7.8) *also hold, then* (7.12) *holds for all $x, y \in H$. Finally, u is the unique viscosity solution of* (7.11) *which belongs to $UC_s([0, T] \times H) \cap C_b([0, T] \times H)$, is B-continuous on $[0, T) \times H$ and satisfies*

$$u(s, x) - g(e^{(T-s)A} x) \to 0 \quad \text{as} \quad s \uparrow T, \text{ uniformly for } x \in B_R \text{ for all } R \geqslant 0. \tag{7.13}$$

Our final result concerns the weak B case in which we will impose the following conditions:

$$\langle b(t, x, \alpha) - b(t, y, \alpha), B(x - y) \rangle \leqslant C_1 \langle B(x - y), x - y \rangle \tag{7.14}$$

for $x, y \in H$, $\alpha \in \mathscr{A}$, $t \in [0, T]$ or

$$\langle b(t, x, \alpha) - b(t, y, \alpha), B(x - y) \rangle \leqslant C_1(R) \langle B(x - y), x - y \rangle \tag{7.15}$$

for $x, y \in B_R$, $\alpha \in \mathscr{A}$, $t \in [0, T]$, where C_1, $C_1(R)$ are positive constants and (7.15) holds for all $R \geqslant 0$. We will also require

$$\left.\begin{array}{l} |f(t, x, \alpha) - f(t, y, \alpha)| \leqslant \omega(\langle B(x-y), x-y\rangle^{1/2}), \\ |g(x) - g(y)| \leqslant \omega(\langle B(x-y), x-y\rangle^{1/2}), \\ \qquad \text{for} \quad x, y \in H, \quad \alpha \in \mathscr{A}, \quad t \in [0, T] \end{array}\right\} \tag{7.16}$$

or

$$\left.\begin{array}{l} |f(t, x, \alpha) - f(t, y, \alpha)| \leqslant \omega_R(\langle B(x-y), x-y\rangle^{1/2}), \\ |g(x) - g(y)| \leqslant \omega_R(\langle B(x-y), x-y\rangle^{1/2}), \\ \qquad \text{for} \quad x, y \in B_R, \quad \alpha \in \mathscr{A}, \quad t \in [0, T] \end{array}\right\} \tag{7.17}$$

for some moduli ω, ω_R, for all $R \geqslant 0$.

THEOREM 7.3. *Let* $(3.5)_w$, (7.2), (7.3), (7.15), (7.17), *and* (7.10) *hold. Then* u *satisfies*

$$\begin{aligned} &|u(s, x) - u(s', y)| \leqslant \omega_R(\|x-y\|_{-1} + |s-s'|), \\ &\textit{for} \quad x, y \in B_R, \qquad s, s' \in [0, T] \end{aligned} \tag{7.18}$$

for some modulus ω_R, *for all* $R \geqslant 0$. *If, in addition,* (7.14) *and* (7.16) *hold then* $u \in UC([0, T] \times H_{-1})$. *Finally,* u *is the unique viscosity solution of* (7.11) *in* $C_b([0, T] \times H)$ *which satisfies*

$$|u(s, x) - u(s, y)| \leqslant \omega_R(\|x-y\|_{-1})$$

for all $R \geqslant 0$, $x, y \in B_R$, $s \in [0, T]$ *for some modulus* ω_R *and* $u(s, x) \to g(x)$ *as* $s \uparrow T$, *uniformly in* $x \in B_R$.

We next want to consider different situations corresponding to "unbounded" controls: indeed, in all the above results, the coefficients are bounded with respect to α at least for x bounded and this excludes control problems such as linear-quadratic control problems. In order to accommodate such cases, we have to introduce some slightly more general setting than the one above. We denote by d the distance on $\mathscr{A}$, recall that we always assume it is a separable metric space, and we (abusing notation) set $d(\alpha) = d(\alpha, \alpha_0)$, where α_0 is a fixed element of $\mathscr{A}$. An admissible control α_t will now be a measurable function from $[0, T]$ into $\mathscr{A}$ such that $d(\alpha_t) \in L^1(0, T)$, where $T > 0$ is given. We still assume that b, f are continuous on $[0, T] \times H \times \mathscr{A}$, b satisfies (7.2), but we now replace (7.3) by

$$\|b(t, x, \alpha)\| \leqslant C_1(1 + \|x\| + d(\alpha)) \qquad \text{for} \quad t \in [0, T], \quad x \in H, \quad \alpha \in \mathscr{A} \tag{7.19}$$

for some positive constant $C_1 \geqslant 0$. Hence, for each admissible control, the state of the system is well defined by (7.1) and $X_s \in C([t, T], H)$. Next, we assume that g, f satisfy (7.9) and

$$\left.\begin{aligned} &|f(t, x, \alpha)| \leqslant C(R), \qquad \|g(x)\| \leqslant C(R) \qquad \text{for} \quad t \in]0, T], \\ &x \in B_R, \quad \alpha \in \mathscr{A}_R = \{\alpha \in \mathscr{A} : d(\alpha) \leqslant R\} \end{aligned}\right\} \tag{7.20}$$

for some positive constant $C(R)$ depending only on $R \in [0, \infty)$, and

$$\begin{aligned} &f(t, x, \alpha) \geqslant -C_2 + \varphi(d(\alpha)) \qquad \text{for} \quad t \in [0, T], \quad x \in H, \quad \alpha \in \mathscr{A} \\ &g(x) \geqslant -C_2 \qquad\qquad\qquad \text{for} \quad x \in H \end{aligned} \tag{7.21}$$

for some positive constant C_2, and some function φ (which we may always assume to be continuous, nonnegative, convex, increasing on $[0, \infty)$) such that $\varphi(r) r^{-1} \to +\infty$ as $r \to +\infty$. Clearly, $u(s, x)$ is well defined on $[0, T] \times H$ by (7.10) and we have on $[0, T] \times H$

$$-C_2(1 + T) \leqslant u(s, x) \leqslant J(s, x, \alpha_0) \leqslant K(\|x\|), \tag{7.22}$$

where α_0 denotes the constant control $\alpha_t = \alpha_0$ for all $t \in [0, T]$ and K is a continuous and increasing function on $[0, \infty)$. But this bound shows that if $R \geqslant 0$ is fixed and $(s, x) \in [0, T] \times B_R$, the infimum in (7.10) may be restricted to those controls α_t which satisfy

$$-C_2 T + \int_s^T \varphi(d(\alpha_t))\, dt \leqslant K(R).$$

Hence, in particular,

$$\int_s^T \varphi(d(\alpha_t))\, dt \leqslant M_R \tag{7.23}$$

and thus

$$\int_s^T d(\alpha_t)\, dt \leqslant C_R \tag{7.24}$$

for some positive constants M_R, C_R. In other words, on balls of H, the control problem is in fact very much similar to those studied before and this may be used to prove the continuity properties of u listed below.

THEOREM 7.4. *We assume* (7.2), (7.19), (7.9), (7.20), (7.21), *and* (7.10). *Then u is bounded from below on $[0, T] \times H$, belongs to $UC_s([0, T] \times B_R) \cap C_b([0, T] \times H)$ for $R \geqslant 0$, and satisfies* (6.4) *restricted to balls of H. Furthermore, u is a solution of* (7.11) *and satisfies $u(T, x) = g(x)$ on H. In addition,* *if* (3.5) *holds, then u satisfies* (7.12). *Finally, if* $(3.5)_w$, (7.15), *and* (7.17) *hold, then u satisfies* (7.18).

Remark 7.5. Let us observe that the Hamiltonian

$$F(t, x, p) = \sup_{\alpha \in \mathscr{A}} \{ -\langle b(t, x, \alpha), p \rangle - f(t, x, \alpha) \}$$

is well defined on $[0, T] \times H \times H$ and satisfies $F \in C_b([0, T] \times H, H)$,

$$\begin{aligned} -C(1 + \|x\|) \|p\| - \Phi(\|x\|) &\leqslant F(t, x, p) \\ &\leqslant C\{1 + (1 + \|x\|) \|p\|\} + \varphi^*(C_1 \|p\|), \end{aligned}$$

where φ^* denotes the convex function dual to φ and Φ is a C^1 positive increasing function on H such that $f(t, x, \alpha_0) \leqslant \Phi(\|x\|)$ on $[0, T] \times H$. In addition, the supremum may be restricted to $\mathscr{A}_M$ for all $t \in [0, T]$, $x \in B_R$, $p \in B_R$, where M is a positive constant depending only on $R \geqslant 0$. From this observation, we deduce easily that F is uniformly continuous in (x, p) bounded, uniformly in $t \in [0, T]$. Moreover, we have

$$|F(t, x, p) - F(t, x, q)| \leqslant C_R \|p - q\| \qquad \text{if} \quad x, p, q \in B_R, \quad t \in [0, T],$$

where C_R is a positive constant depending only on $R \geqslant 0$. Finally, we have also

$$\begin{aligned} &|F(t, x, p) - F(t, y, p)| \leqslant C_0 \|x - y\| \, \|p\| + \omega_R(\|x - y\|) \\ &\qquad \text{for} \quad t \in [0, T], \quad x, y \in B_R, \quad p \in H, \end{aligned}$$

and if (7.15), (7.17) hold

$$\begin{aligned} &|F(t, x, \lambda B(x - y)) - F(t, y, \lambda B(x - y))| \\ &\quad \leqslant C_R \lambda \|x - y\|_{-1}^2 + \omega_R(\|x - y\|) \qquad \text{for} \quad t \in [0, T], \quad x, y \in B_R, \quad \lambda \geqslant 0. \end{aligned}$$

We now turn to uniqueness statements.

THEOREM 7.6. *Let* (7.2), (7.19), (7.9), (7.20), (7.21), *and* (7.10) *hold.*

(i) (*Strong B case*) *If, in addition,* (3.5) *holds and* $v, w \in UC_s([0, T] \times B_R) \cap C_b([0, T] \times H)$ *for* $R \geqslant 0$ *are, respectively, a sub and a supersolution of* (7.11) *satisfying*

$$\begin{aligned} &\lim_{t \uparrow T} \big((v(t, x) - v(T, e^{-A(T-t)} x))^+ \\ &\qquad + (w(t, x) - w(T, e^{-A(T-t)} x))^- \big) = 0 \qquad \textit{uniformly on } B_R \qquad (*) \end{aligned}$$

for $R \geqslant 0$, w is bounded from below on $[0, T] \times H$, and v, w are B-continuous on $[0, T) \times H$, then

$$\sup_{x \in H} \{v(t, x) - w(t, x)\} \leqslant \sup_{x \in H} \{v(T, x) - w(T, x)\}$$

for $t \in [0, T]$. In particular, the value function u is the unique solution of (7.11) *in $UC_s([0, T) \times B_R) \cap C_b([0, T] \times H)$ which is bounded from below, B-continuous on $[0, T) \times H$, and satisfies $u(T, x) = g(x)$ on H and $(*)$ with $v \equiv w \equiv u$.*

(ii) (*Weak B case*) *If, in addition,* $(3.5)_w$, (7.15), *and* (7.17) *hold and $v, w \in C_b([0, T] \times H)$ are, respectively, a sub and supersolution of* (7.11) *satisfying*

$$|\varphi(t, x) - \varphi(t, y)| \leqslant \omega_R(\|x - y\|_{-1})$$
$$\text{for} \quad \varphi = v, w, \quad t[0, T], \quad x, y \in B_R \tag{7.25}$$

and some modulus ω_R depending on $R \geqslant 0$,

$$\lim_{t \uparrow T} ((v(t, x) - v(T, x))^+ + (w(t, x) - w(T, x))^-) = 0$$
$$\text{uniformly in } x \in B_R, \text{for } R \geqslant 0, \tag{7.26}$$

and w is bounded from below on $[0, T] \times H$, then (7.24) *holds. In particular, the value function u is the unique solution of* (7.11) *in $C_b([0, T] \times H)$ which is bounded from below, satisfies* (7.25), (7.26) *with $v \equiv w \equiv u$ and $u(T, x) = g(x)$ on H.*

Sketch of Proof. We only sketch the proof of the above result in the strong B case, the weak B case being treated in a similar fashion. Without loss of generality, we may always assume that $v(T, x) \leqslant w(T, x)$ on H and we then choose $g(x) = w(T, x)$. Therefore, we only have to prove that

$$v(t, x) \leqslant u(t, x) \leqslant w(t, x) \qquad \text{on} \quad [0, T] \times H. \tag{7.27}$$

We first prove the left part of the above chain of inequalities. A simple proof consists in introducing the value functions u_n corresponding to the same control problem as the one defining u but with $\mathscr{A}$ replaced by $\mathscr{A}_n$ $(= \{\alpha \in \mathscr{A} \mid d(\alpha) \leqslant n\})$. By standard density arguments, one checks that $u_n(t, x) \downarrow u(t, x)$ on $[0, T] \times H$, as $n \uparrow \infty$. But, v is still a subsolution of the corresponding HJB equation, i.e., (7.11) with $\sup_{\alpha \in \mathscr{A}}$ replaced by $\sup_{\alpha \in \mathscr{A}_n}$. We may now apply the comparison results in the bounded case and deduce $v \leqslant u_n$ on $[0, T] \times H$. We conclude letting n go to ∞.

The proof of the second inequality of (7.27) is more elaborate. One possible argument consists in remarking first that, without loss of generality, we may assume that $f \geqslant 1 + \varphi(d(\alpha))$ on $[0, T] \times H \times \mathscr{A}$ (add $(1 + C_2)(T - t)$ to u and w). Then, we introduce

$$V = 1 - e^{-u}, \qquad W = 1 - e^{-w} \qquad \text{on} \quad [0, T] \times H.$$

Clearly, V and W are bounded on H and by standard "viscosity manipulations" are, respectively, a sub and a supersolution of

$$\begin{aligned} -\frac{\partial z}{\partial s} + \langle Ax, \nabla z \rangle + \sup_{\alpha \in \mathscr{A}} \{ -\langle b(s, x, \alpha), \nabla z \rangle \\ + f(s, x, \alpha) z - f(s, x, \alpha) \} = 0 \qquad \text{in} \quad (0, T) \times H. \end{aligned} \tag{7.28}$$

The next step consists in perturbing W in order to reduce the comparison between u and w (or, equivalently, V and W) to a local comparison. We thus want to choose $\psi \in C^1([0, \infty))$ increasing on $[0, \infty)$ such that $\psi(0) = 1$, $\psi'(0) = 0$, $\psi(r) \to \infty$ as $r \to \infty$ and $W + \gamma \psi(\|x\|) e^{-Ks}$ is still a supersolution of (7.28) for all $\gamma > 0$ and some $K > 0$. In view of the definition of solutions, it is enough to ask that

$$\begin{aligned} \inf_{\alpha \in \mathscr{A}} \{ f(s, x, \alpha) \psi - \langle b(s, x, \alpha), \nabla \psi \rangle \} \\ \text{is bounded from below on } [0, T] \times H, \end{aligned} \tag{7.29}$$

and we just have to choose $\psi(x) = 1 + \log(1 + \|x\|^2)$ remarking that

$$f(s, x, \alpha) \psi - \langle b(s, x, \alpha), \nabla \psi \rangle \geqslant \varphi(d(\alpha)) - 3C_1 - 2C_1 d(\alpha) \geqslant -K$$

for all $x \in H$, $t \in [0, T]$, $\alpha \in \mathscr{A}$, for some $K \geqslant 0$.

We next observe that, for each $\gamma > 0$, there exists $R = R(\gamma) \geqslant 0$ such that $V \leqslant W + \gamma \psi e^{-Ks}$ for $\|x\| \geqslant R$, $t \in [0, T]$ and that

$$\psi(e^{-A(T-t)} x) \leqslant \psi(x) \qquad \text{for all} \quad x \in H, \quad t \in [0, T].$$

Therefore, in view of the properties of the Hamiltonian listed in Remark 7.5, we deduce from the comparison results and their proofs

$$V \leqslant W + \gamma \psi e^{-Ks} \qquad \text{on} \quad [0, T] \times B_R, \qquad \text{for all} \quad R \geqslant 0, \quad \gamma > 0$$

and we obtain

$$V \leqslant W + \gamma \psi e^{-Ks} \qquad \text{on} \quad [0, T] \times H, \qquad \text{for all} \quad \gamma > 0.$$

We conclude letting $\gamma \downarrow 0$.

VIII. Weakly Continuous Solutions Revisited and Miscellaneous Remarks

As we recalled several times before, the main difference between the results proven in the preceding sections and those given in [12, Part IV] is the replacement of (sequentially) weakly continuous solutions by B-continuous solutions. Of course, when B is compact, as it was the case in the existence results of [12, Part IV], there is no difference and B-continuous solutions are indeed weakly continuous. However, in general, this is not the case (take $A=0$, $B=I$ in which case B-continuity is nothing but continuity in H for the norm topology) and the weak continuity is then a qualitative property (or regularity property) enjoyed by some solutions. In particular, we give in this section a result which shows how the (sequential) weak continuity of solutions u may a consequence of the weak continuity of data; that is, of the Hamiltonian (for (S)) and the Hamiltonian and the initial condition (for (E)).

Since there are infinitely many results of this sort, we just give examples in the case of bounded solutions. We use the notation.

$$x_n \rightharpoonup x$$

to denote the weak convergence of a sequence x_n to x in H.

We begin with the case of (E).

THEOREM 8.1. *Let* (3.3) *hold as well as*

$$x_n \rightharpoonup x \textit{ implies } F(t, x_n, p) \to F(t, x, p) \textit{ uniformly in } p \in B_R,\ t \in [0,], \tag{8.1}$$

for all $R \geqslant 0$ *and*

$$\psi \textit{ is weakly sequentially continuous on } H. \tag{8.2}$$

(i) (*Strong B case*) *In addition, let* (3.1), (3.5), (6.3) *hold and* $\psi \in UC(H)$. *Then, the unique solution of* (E) *given by Theorem* 6.1 *is sequentially weakly continuous on* $[0, T] \times H$.

(ii) (*Weak B case*) *In addition, let* $(3.5)_w$, (3.9) *hold*, $\psi \in UC(H_{-1})$, *and* $F(t, x, 0)$ *be bounded on* $[0, T] \times H$. *Then the unique solution of* (E) *given by Theorem* 6.4 *is sequentially weakly continuous on* $[0, T] \times H$.

Remark 8.2. As we said above, there are infinitely many variants of results of this sort. We mention another which will be useful in the sequel; namely, instead of (6.3), we may merely assume that $F(t, x, 0)$ is bounded on $[0, T] \times H$ in the strong B case if $\psi = 0$ ($\pm Ct$ are then super and sub solutions for C large enough, which takes care of the difficulty of the initial layer...).

Proof of Theorem 8.1. We sketch the proof in the weak B case, the proof in the strong B case being basically the same. Let $R_0 \in (0, \infty)$ be fixed and let $h \in B_{R_0}$, let $t_0 \in (0, T]$, we consider a new function defined on $[0, t_0] \times H$ by $\tilde{u}(t, x) = u(t, x + e^{-(t_0 - t)A^*}h)$. And we wish to compare u and $\tilde{u}$. In order to do so, we first observe that by the arguments of Section II, $\tilde{u}$ is a solution on $(0, t_0) \times H$ of

$$\frac{\partial \tilde{u}}{\partial t} + \langle Ax, \nabla \tilde{u} \rangle + F(t, x + e^{-(t_0 - t)A^*}h, \nabla u) = 0. \tag{8.3}$$

Next, we introduce a distance on H which yields the weak topology on balls: indeed, let $\{e_n\}_{n \geqslant 1}$ be an orthonormal basis of H and set

$$d(x, y) = \sum_{n \geqslant 1} \frac{1}{2^n} \|(x - y, e_n)\|, \qquad \text{for all} \quad x, y \in H.$$

And we observe that (8.1), (8.2) amount to requiring that for all $R \geqslant 0$

$$|F(t, x, p) - F(t, y, p)| \leqslant \omega_R(d(x, y)), \qquad \|\psi(x) - \psi(y)\| \leqslant \omega_R(d(x, y)) \tag{8.4}$$

for all $x, y, p \in B_R$, $t \in [0, T]$ for some modulus ω_R. Observe also that, if we set $d(x) = d(0, x)$,

$$\sup_{0 \leqslant t \leqslant t_0 \leqslant T} d(e^{-(t_0 - t)A^*}h) \leqslant \gamma(d(h)) \tag{8.5}$$

for some modulus γ depending only on R_0 (recall that $h \in B_{R_0}$).

Once all these remarks are made, we perform the uniqueness proofs in order to find for all $t \in [0, t_0]$, $x, y \in H$

$$u(t, x) - \tilde{u}(t, y) \leqslant \delta\{\mu(x) + \mu(y)\} + \frac{1}{2\varepsilon} \langle B(x - y), x - y \rangle$$
$$+ m_\varepsilon(\delta) + K(\varepsilon) + \omega_{\varepsilon, \delta}(d(h)), \tag{8.6}$$

where $\omega_{\varepsilon, \delta}$ is a modulus depending only on $\varepsilon, \delta \in (0, 1]$ and R_0, m_ε is a modulus depending only on ε and K is a modulus. In particular, all these quantities are independent of $t_0 \in [0, T]$. Then, (8.6) yields easily for all $R \geqslant 0$

$$u(t_0, x) - u(t_0, x + h) \leqslant m(d(h)), \qquad \text{for all} \quad t_0 \in [0, T], \quad x \in B_R, \quad h \in B_{R_0}$$

for some modulus depending only on R and R_0. Therefore, we have for all $R_0 > 0$

$$\|u(t, x) - u(t, y)\| \leqslant m_{R_0}(d(x, y)), \qquad \text{for all} \quad t \in [0, T], \quad x, y \in B_{R_0}$$

and this proves Theorem 8.1.

We now turn to the case of (S).

THEOREM 8.3. *Let* (3.3), (8.1) *hold and* $F(x, 0)$ *be bounded on* H.

(i) (*Strong* B *case*) *If, in addition,* (3.1) *and* (3.5) *hold, then the unique solution of* (S) *given by Theorem* 3.1 *is sequentially weakly continuous on* H.

(ii) (*Weak* B *case*) *If, in addition,* $(3.5)_w$ *and* (3.9) *hold, then the unique solution of* (S) *given by Theorem* 3.7 *is sequentially weakly continuous on* H.

The proof of Theorem 8.3 is a straightforward consequence of Theorem 8.1: indeed, we approximate (S) by

$$\frac{\partial u}{\partial t} + \langle Ax, \nabla u \rangle + F(x, \nabla u) + u = 0 \qquad \text{in} \quad (0, \infty) \times H$$

with the initial condition $u|_{t=0} = 0$ on H. By Remark 8.2, the addition of u introducing no changes in the preceding proofs, we can apply Theorem 8.1 and we find that $u(t, x)$ is sequentially weakly continuous on $[0, \infty) \times H$. We conclude easily since, by comparison results,

$$\sup_{x \in H} |u(x) - u(t, x)| \leqslant e^{-t} \sup_{z \in H} |u(z)|,$$

where u is the solution of (S).

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